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Integral Equations
and Operator Theory

COMPACT LINEAR OPERATORS ON FUNCTIONAL SPACES WITH TWO NORMS

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Some principles of the operator theory in a linear space with two norms are established in this paper. The well-known Hilbert-Schmidt theorem on the eigenfunction expansion of sourcewise represented functions, Mercer's theorem and other results can be consider as special cases of the statements presented. The general approach proposed is used to construct the theory of symmetrizable operators and to investigate the asymptotic behaviour of eigenvalues of compact operators.

It is known that a number of statements of the Hilbert-Schmidt theory of integral equations with a real symmetric kernel can be obtained as a simple consequence of the Hilbert theory of compact Hermitian operators on a Hilbert space H. At the same time, some important assertions of the former theory cannot in principle be derived from the operator theory in an abstract Hilbert space. In fact, in the language of functional spaces, the convergence in H means the convergence in the mean, whereas in the Hilbert-Schmidt theory the uniform convergence of certain series is claimed. For the abstract operator theory to cover all the results of the Hilbert-Schmidt theory of integral equations, it is necessary to consider operators on H where, in addition to the original norm (the square root of the scalar square of an element), a new Riesz-Banach norm is introduced for a certain set of elements. For continuous functions this norm corresponds to the uniform one. It is possible to solve the above problem by investigating the convergence of spectral expansions with respect to the new norm. In what follows, our results in this direction are expounded.

In Section 1 we establish some general facts of the linear operator theory in a space with two norms. The well-known Hilbert-Schmidt theorem on expansions of so-called "quellenmässige" functions in fundamental functions, Mercer's theorem, etc. are interpretations of these general assertions. Other interpretations enabled us to develop (Section 2) the symmetrizable (Hermitizable) operator theory. In Section 3 we present the theorem which enables us to investigate the asymptotics of eigenvalues of a compact operator. The application of our results to the theory of integral equations with a symmetrizable kernel as well as to the discovery of the asymptotics of characteristic numbers of integral equations will be given elsewhere.

Theorems 1 - 4 were presented at the USSR Academy of Sciences Session on Functional Analysis (August, 1937).

1. Principal Theorems

1. Let L be a linear space satisfying all the axioms of a complex Hilbert space except that of completeness. As usual, we denote by (g, f) the scalar product of the elements $g, f \in L$. A linear operator A defined on the whole L is called Hermitian if

$$(Ag, f) = (g, Af) \quad (g, f \in L).$$

Lemma 1 *Let a functional $p(f)$ given on L possess the property*

$$\|f\| = \sqrt{(f, f)} \leq p(f) \quad (f \in L). \tag{1.1}$$

If a Hermitian operator A on L satisfies the condition

$$p(Af) \leq p(f) \quad (f \in L), \tag{1.2}$$

then its norm does not exceed 1, *i.e.*

$$(Af, Af) \leq (f, f) \quad (f \in L). \tag{1.3}$$

Proof. For an element $f \in L$ we put

$$q = \frac{(Af, Af)}{(f, f)}, \quad s_n = (A^n f, A^n f) \quad (n = 0, 1, 2, \dots).$$

The equality

$$(A^{n-1} f + \lambda A^{n+1} f, A^{n-1} f + \lambda A^{n+1} f) = s_{n-1} + 2\lambda s_n + \lambda^2 s_{n+1} \geq 0 \quad (n = 1, 2, \dots)$$

valid for all real λ implies

$$s_n^2 \leq s_{n-1} s_{n+1} \quad (n = 1, 2, \dots).$$

Since

$$s_n \geq 0 \quad (n = 0, 1, 2, \dots),$$

we have

$$q = \frac{s_1}{s_0} \leq \frac{s_2}{s_1} \leq \frac{s_3}{s_2} \leq \dots,$$

whence

$$s_n \geq s_0 q^n \quad (n = 1, 2, \dots); \quad q \leq \liminf \sqrt[n]{s_n}.$$

On the other hand, according to (1.1) and (1.2),

$$s_n = (A^n f, A^n f) \leq p^2(A^n f) \leq p^2(f),$$

hence

$$\limsup \sqrt[n]{s_n} \leq 1.$$

Therefore, $q \leq 1$, which yields (1.3). This completes the proof.

2. In the sequel we suppose that in L a Riesz-Banach norm $\|f\|_B$ is introduced, not equivalent to the Hilbert norm

$$\|f\|_H = \sqrt{(f,f)} \quad (f \in L)$$

So,

1) $\|f\|_B > 0$ if $f \neq 0$;

2) $\|f+g\|_B \leq \|g\|_B + \|f\|_B$ $(g, f \in L)$;

3) $\|\lambda g\|_B = |\lambda| \, \|g\|_B$ $(g, f \in L,\ \lambda$ is a scalar).

It is assumed, in addition (this corresponds to the fact that the uniform convergence of a sequence of continuous functions causes its mean convergence), that if $\{f_n\}, f_n \in L$, is a Cauchy sequence with respect to $\|\cdot\|_B$, i.e.

$$\|f_n - f_m\|_B \to 0 \quad \text{as } m, n \to \infty,$$

then it is a Cauchy sequence with respect to the norm $\|\cdot\|_H$, i.e.

$$\|f_n - f_m\|_H \to 0 \quad \text{as } m, n \to \infty.$$

As is known, this condition is equivalent to the existence of a real constant C such that

$$\|f\|_H \leq C \, \|f\|_B \quad (f \in L). \tag{1.4}$$

Without loss of generality, we may put $C = 1$, i.e.

$$(f,f) \leq \|f\|_B^2 \quad (f \in L). \tag{1.5}$$

Denote by L_B the linear space L with the norm $\|\cdot\|_B$. If we consider L as a Hilbert (generally, incomplete) space with the scalar product $(\cdot,\cdot)$, we denote it by L_H. A linear operator on L is called H-continuous (B-continuous) if it is continuous in L_H (respectively, in L_B). Analogously, the terms H-bounded (B-bounded), H-convergent (B-convergent), etc. are used.

Theorem 1 *If a Hermitian operator A $(AL \subset L)$ is B-continuous, then A is H-continuous too. Moreover,*

$$\|A\|_H = \sup_{f \in L} \frac{\|Af\|_H}{\|f\|_H} \leq \sup_{f \in L} \frac{\|Af\|_B}{\|f\|_B} = \|A\|_B. \tag{1.6}$$

Proof. Let

$$A_1 = \frac{1}{\|A\|_B} A, \quad p(f) = \|f\|_B \quad (f \in L).$$

By (1.5), the conditions of Lemma 1 hold true for the operator A_1 and the functional $p(f)$, whence

$$\|A_1 f\|_H^2 = (A_1 f, A_1 f) \leq \|f\|_H^2. \quad \cdot$$

which is equivalent to (1.6). The proof is complete.

3. For the sake of simplicity, in future we presume the space L_B to be complete. Certainly, in this case L_H is not complete, for otherwise, by the well-known Banach theorem (see [1], Section 3), the B- and the H-norms would be topologically equivalent on L. We may consider L as a dense part of a certain complete Hilbert space denoted by $\mathfrak{H}$. Evidently, an H-continuous linear operator A admits a unique continuous (linear) extension $\widetilde{A}$ to the whole $\mathfrak{H}$.

Recall that a complex number λ is called a regular point of a continuous linear operator A on a complete linear normed space E ($AE \subset E$) if the map $f \mapsto (A - \lambda I)f$ is one-to-one from E onto E. All the nonregular points of the operator A form its spectrum. We denote by $\sigma_B(A)$ the spectrum of a linear operator A ($AL \subset L$) continuous in L_B, and by $\sigma_H(A)$ that of the continuous extension $\widetilde{A}$ to $\mathfrak{H}$.

Theorem 2 *(Theorem 2 is also true for a closed operator with a dense domain in L_B if we define its spectrum appropriately.) Let A ($AL \subset L$) be a B-continuous Hermitian operator. Then*

$$\sigma_H(A) \subset \sigma_B(A).$$

Proof. Since the continuous extension $\widetilde{A}$ ($\widetilde{A}\mathfrak{H} \subset \mathfrak{H}$) of the Hermitian operator A is also Hermitian, the spectrum $\sigma_H(A)$ is real. It remains to prove that a real point $\lambda \notin \sigma_B(A)$ does not belong to $\sigma_H(A)$ either. But if $\lambda \notin \sigma_B(A)$, $\Im\lambda = 0$, then the B-continuous operator $R_\lambda = (A - \lambda I)^{-1}$ exists. By Theorem 1, the Hermitian operator R_λ is H-continuous, hence $\lambda \notin \sigma_H(A)$. This completes the proof.

4. In accordance with the B- and H-terminologies, we call a linear operator A ($AL \subset L$) B-compact (H-compact) if for any set $E \subset L$ bounded in L_B (in L_H), the set AE is compact in L_B (conditionally compact[2] in L_H).

If an operator A is B-compact, then, by the Riesz theorem [2], its spectrum $\sigma_B(A)$ consists of zero[3] and the set $\{\lambda^{(j)}\}$ of the eigenvalues $\lambda^{(j)} \neq 0$ of A. The set $\{\lambda^{(j)}\}$ is either a finite one or an infinite sequence approaching zero. Moreover, the linear set $E^{(j)}$ of the vectors $\varphi \in L$ for each of which there exists a natural n such that

$$\left(A - \lambda^{(j)} I\right)^n = 0 \tag{1.7}$$

is finite-dimensional. The number $n_j = \dim E^{(j)}$ is the multiplicity of the eigenvalue $\lambda^{(j)}$. If, in addition, the operator A is Hermitian, then $E^{(j)}$ consists of its eigenvectors φ ($A\varphi = \lambda^{(j)}\varphi$) only. Indeed, since $AE^{(j)} = E^{(j)}$ and the operator A is Hermitian, there exists an orthonormal basis consisting of eigenvectors of A. In view of (1.7), each of these eigenvectors corresponds to the eigenvalue $\lambda^{(j)}$.

Theorem 3 *Every B-compact Hermitian operator A ($AL \subset L$) is H-compact; moreover, $\sigma_H(A)$ coincides with $\sigma_B(A)$, and the multiplicities of an eigenvalue $\lambda \neq 0$ in L_B and in L_H are the same.*

[2]The difference between the definitions of the B- and the H-compactness of the operator A is caused by the fact that, in contrast to L_B, L_H is not complete.

[3]The space L is assumed to be infinite-dimensional; in the opposite case all the theorems are trivial.

Proof. As we already know, the spectrum $\sigma_B(A)$ consists of zero and the sequence $\{\lambda^{(j)}\}$. It is also obvious that $\lambda^{(j)} \in \sigma_H(A)$. Let $E^{(j)}$ be as above. Denote by $L^{(j)}$ the set of all $f \in L$ orthogonal to $E^{(j)}$. As the operator A is Hermitian, $L^{(j)}$ is invariant for A. We shall show that the space $L_B^{(j)}$ is complete.

Assume that a sequence $\{g_n\} \subset L^{(j)}$ B-converges to $g \in L_B$. Then

$$(\varphi, g_n) = 0 \quad (n = 1, 2, \dots)$$

for any $\varphi \in E^{(j)}$. By (1.5),

$$|(\varphi, g)| = |(\varphi, g - g_n)| \le \|\varphi\|_H \|g - g_n\|_H \le \|\varphi\|_H \|g - g_n\|_B \quad (n = 1, 2, \dots),$$

whence $(\varphi, g) = 0$ because $\|g - g_n\|_B \to 0$ when $n \to \infty$. So $g \in L^{(j)}$, which proves the completeness of $L_B^{(j)}$.

Consider A as a compact operator on $L_B^{(j)}$. The point $\lambda^{(j)}$ does not belong to the spectrum of this operator, for otherwise the equation $A\varphi - \lambda^{(j)}\varphi = 0$ would have a solution $\varphi \neq 0$, contrary to the fact that $E^{(j)}$ is the set of all the solutions of this equation. But then $\lambda^{(j)}$ does not belong to the spectrum of A considered as an operator on $L_H^{(j)}$. Consequently, the continuous in $L_H^{(j)}$ operator $(A - \lambda^{(j)} I)^{-1}$ exists.

Now let the space $\mathfrak{H}$ and the operator $\widetilde{A}$ be as in Subsection 3. Denote by $\mathfrak{H}^{(j)}$ the orthogonal complement to $E^{(j)}$ in $\mathfrak{H}$. Evidently, $L_H^{(j)}$ is dense in $\mathfrak{H}^{(j)}$. Therefore, the continuous operator $B^{(j)} = (A - \lambda^{(j)} I)^{-1}$ given on $L_H^{(j)}$ admits a unique continuous extension $\widetilde{B}^{(j)}$ to $\mathfrak{H}^{(j)}$ which is also Hermitian. Since the operators $A - \lambda^{(j)} I$ and $B^{(j)}$ are each other's inverses, so are the operators $\widetilde{A} - \lambda^{(j)} I$ and $\widetilde{B}^{(j)}$. Thus, we have proved that $E^{(j)}$ is the whole eigensubspace of the operator $\widetilde{A}$ corresponding to the eigenvalue $\lambda^{(j)}$ and $\lambda^{(j)}$ does not belong to the continuous spectrum[1] of $\widetilde{A}$. Insofar as every regular point of the operator A on L_H is that of the operator $\widetilde{A}$ on $\mathfrak{H}$ and vice versa, the spectra of A and $\widetilde{A}$ are the same. Taking into account that, by Theorem 2, $\sigma_H(A) \subset \sigma_B(A)$, we conclude that the spectrum of $\widetilde{A}$ consists of the sequence of finite multiplicity eigenvalues $\{\lambda^{(j)}\}$ and, possibly, zero. Since the spectrum is always closed, $0 \in \sigma_B(A)$, for otherwise the sequence $\{\lambda^{(j)}\}$ as well as $\dim L = \sum n_j$ would be finite, which contradicts our assumption. So $\sigma_B(A) = \sigma_H(A)$.

The above-established structure of the spectrum $\sigma_H(A)$ of the Hermitian operator $\widetilde{A}$ is, as we know, characteristic for a compact operator. Therefore, the operator $\widetilde{A}$, and hence the operator A, is compact. The proof is complete.

It should be noted that Theorems 1,2 are not true for non-Hermitian operators. In fact, as the norms $\|\cdot\|_B$ and $\|\cdot\|_H$ are not topologically equivalent, there exists a linear functional $f_0(x)$ continuous on L_B but discontinuous on L_H. Take an arbitrary element $a \in L$ and put $Ax = f_0(x)\, a$ $(x \in L)$. It is clear that the operator A is B- but not H-continuous (the example is due to S.G.Krein).

5. In every subspace $E^{(j)}$ we choose an orthonormal basis. The union of all these bases gives the orthonormal system $\{\varphi_j\}$ of eigenvectors of A $(\widetilde{A})$. We call the system $\{\varphi_j\}$ fundamental. Let $\{\lambda_j\}$ be the sequence of the corresponding eigenvalues of A:

$$A\varphi_j = \lambda_j \varphi_j \quad (j = 1, 2, \dots).$$

[1] We refer to [3] for the classification of the spectrum of a Hermitian operator.

For any $g \in \mathfrak{H}$, the spectral expansion

$$\widetilde{A}g = \sum_{j=1}^{\infty} \lambda_j (g, \varphi_j)\varphi_j \tag{1.8}$$

converges in $\mathfrak{H}$.

In what follows, only the case where the system $\{\varphi_j\}$ is infinite is of interest. The equality (1.8) implies in particular

$$Ag = \sum_{j=1}^{\infty}{}_H \lambda_j (g, \varphi_j)\varphi_j \quad (g \in L). \tag{1.9}$$

The subscript H in the sum indicates that the expansion (1.9) is convergent in L_H (H-convergent). As for its B-convergence, either of the cases is possible.

If A ($AL \subset L$) is an arbitrary H-compact Hermitian operator, then extending it continuously to a compact Hermitian operator $\widetilde{A}$ on $\mathfrak{H} \supset L$ ($\overline{L_H} = \mathfrak{H}$) and taking its fundamental orthonormal system of eigenvectors $\{\varphi_j\}$, we shall again have the expansions (1.8) and (1.9). However, in this general case, we cannot guarantee that the vectors $\varphi_j \in \mathfrak{H}$ belong to L. Below, we shall describe the class of H-compact operators A such that $\{\varphi_j\} \subset L$ and the expansion (1.9) is B-convergent.

Definition 1 *We say that a linear operator A ($AL \subset L$) is HB-continuous if it transforms each H-bounded set from L into a B-bounded one.*

It is easy to see that an HB-continuous operator may be defined as an operator for which there exists a constant C such that

$$\|Ag\|_B \leq C\,\|g\|_H \quad (g \in L). \tag{1.10}$$

As a B-bounded set is always H-bounded, an HB-continuous operator is H- and B-continuous simultaneously.

Denote by L_B^* the dual of L_B, i.e. the space of all continuous linear functionals Φ on the whole L_B. If $\Phi \in L_B^*$, then

$$\|\Phi\| = \sup_{g \in L} \frac{|\Phi(g)|}{\|g\|_B} < \infty.$$

By (1.10),

$$|\Phi(Ag)| \leq \|\Phi\| \cdot \|Ag\|_B \leq C\,\|\Phi\| \cdot \|g\|_H.$$

This shows that $\Phi(Ag)$ is a continuous linear functional on L_H. Hence, a certain element $a_\Phi \in \mathfrak{H}$ corresponds to this functional so that

$$\Phi(Ag) = (g, a_\Phi) \quad (g \in L). \tag{1.11}$$

We have proved the necessary condition of the following HB-continuity criterion.

1°. *A linear operator A ($AL \subset L$) is HB-continuous if and only if an element $a_\Phi \in \mathfrak{H}$ can be assigned to each functional $\Phi \in L_B^*$ so that the equality (1.11) holds.*

The sufficiency of the condition follows from the inequality

$$|\Phi(Ag)| \leq \|g\|_H \cdot \|a_\Phi\|$$

which is a consequence of (1.11). It implies that for every functional $\Phi \in L_B^*$, the set $|\Phi(Ag)|$, $g \in S$, is bounded where S is the unit H-ball ($\|g\|_H \leq 1$); hence the set AS is B-bounded.

Now, we shall verify that the closure $\widetilde{A}$ in $\mathfrak{H}$ of an HB-continuous operator A ($AL \subset L$) possesses the property

$$\widetilde{A}\mathfrak{H} \subset L.$$

Indeed, for $g \in \mathfrak{H}$, there exists a sequence $\{g_n\} \subset L$ such that $g_n \to g$ in $\mathfrak{H}$, so

$$Ag_n \to \widetilde{A}g \quad (\text{in } \mathfrak{H}). \tag{1.12}$$

On the other hand, in view of (1.10),

$$\|Ag_n - Ag_m\|_B \leq C\,\|g_n - g_m\|_H \quad (n, m = 1, 2, \dots),$$

hence $\{Ag_n\}$ is a Cauchy sequence in L_B. Since the space L_B is complete, there exists $f \in L$ such that Ag_n B-converges to f. But then Ag_n H-converges to f, i.e. $Ag_n \to f$ in $\mathfrak{H}$. Taking into account (1.12), we have

$$\widetilde{A}g = f \in L.$$

Therefore, if the operator A is HB-continuous and φ is an eigenvector of its closure $\widetilde{A}$, i.e. $\widetilde{A}\varphi = \lambda\varphi$, then

$$\varphi = \lambda^{-1}\widetilde{A}\varphi \in L,$$

provided that $\lambda \neq 0$.

To prove Theorem 4, we need the next lemma.

Lemma 2 [1] *Let E be a complete linear normed space, and $p(x)$ a nonnegative functional on the whole E which has the properties:*

1. $p(x + y) \leq p(x) + p(y)$;
2. $p(\lambda x) = |\lambda|\, p(x)$

($x, y \in E$, λ is a scalar). In order for the functional $p(x)$ to be continuous, it is sufficient for it to be lower semicontinuous.

As is easily seen, the continuity of the functional $p(x)$ satisfying the above two conditions is equivalent to the existence of a constant $\gamma > 0$ such that $p(x) \leq \gamma\|x\|$ ($x \in E$).

[1] For the first time, a statement of such a kind was established by Banach. Generalizing Banach's result, Orlicz [4] proved the assertion which contains Lemma 2 as a specific case. In the form given here, Lemma 2 can be found in Gel'fand's paper [5].

Theorem 4 *If an H-compact Hermitian operator A ($AL \subset L$) is HB-continuous, then its spectral expansion B-converges, i.e.*

$$Ag = \sum_{j=1}^{\infty}{}_{B} \lambda_j(g, \varphi_j)\,\varphi_j \quad (g \in L). \tag{1.13}$$

Proof. Let (1.9) be the spectral expansion of the operator A in L_H. For $g \in L$ we set

$$f_n = \sum_{j=1}^{n} \lambda_j(g, \varphi_j)\,\varphi_j = \sum_{j=1}^{n} (g, \varphi_j)\,A\varphi_j \quad (n = 1, 2, \dots).$$

Then for an arbitrary $\Phi \in L_B^*$,

$$|\Phi(f_m) - \Phi(f_n)| = \Big|\sum_{j=m}^{n} (g, \varphi_j)\,\Phi(A\varphi_j)\Big| = \Big|\sum_{j=m}^{n} (g, \varphi_j)(\varphi_j, a_\Phi)\Big| \le$$

$$\Big\{\sum_{j=m}^{n} |(g, \varphi_j)|^2 \cdot \sum_{j=m}^{n} |(\varphi_j, a_\Phi)|^2\Big\}^{1/2} \le p(\Phi)\Big\{\sum_{j=m}^{n} |(g, \varphi_j)|^2\Big\}^{1/2} \quad (m, n = 1, 2, \dots), \tag{1.14}$$

where

$$p(\Phi) = \Big\{\sum_{j=1}^{\infty} |\Phi(A\varphi_j)|^2\Big\}^{1/2} \le (a_\Phi, a_\Phi)^{1/2}.$$

It is not hard to see that the functional $p(\Phi)$ satisfies all the conditions of Lemma 2. The properties 1 and 2 are verified directly. Furthermore, $p(\Phi)$ is a limit of the nondecreasing sequence of continuous functionals

$$p_n(\Phi) = \Big\{\sum_{j=1}^{n} |\Phi(A\varphi_j)|^2\Big\}^{1/2}.$$

So $p(\Phi)$ is lower semicontinuous.

Consequently, there exists a positive constant γ such that

$$p(\Phi) \le \gamma\,\|\Phi\| \quad (\Phi \in L_B^*). \tag{1.15}$$

Choose $\Phi \in L_B^*$ so that

$$\|\Phi\| = 1, \quad \Phi(f_m - f_n) = \|f_m - f_n\|_B.$$

By (1.14) and (1.15),

$$\|f_m - f_n\|_B \le \gamma\Big\{\sum_{j=m}^{n} |(g, \varphi_j)|^2\Big\}^{1/2} \quad (m, n = 1, 2, \dots).$$

Since

$$\sum_{j=1}^{\infty} |(g, \varphi_j)|^2 \le (g, g),$$

the sequence $\{f_n\}$ B-converges. Let $f \in L_B$ be its limit. Moreover, $f_n \to f$ in L_H. But in L_H, according to (1.9), $f_n \to Ag$. This implies $f = Ag$. Thus, $f_n \to Ag$ in L_B, which is what had to be proved.

6. In the space L_B^* of continuous linear functionals on L_B we define the multiplication by a scalar λ in an unusual way. We write

$$\Psi = \lambda\Phi \quad (\Phi \in L_B^*,\ \lambda \text{ is a scalar})$$

if

$$\Psi(g) = \overline{\lambda}\,\Phi(g)$$

for any $g \in L$. As for the addition in L_B^*, the ordinary definition is preserved. As a convention we also write

$$\Phi(g) = (g, \Phi) = \overline{(\Phi, g)} \quad (g \in L,\ \Phi \in L_B^*).$$

In accordance with this agreement, the equality (1.11) for an HB-continuous operator A takes the form

$$(Ag, \Phi) = (g, a_\Phi) \quad (g \in L,\ \Phi \in L_B^*).$$

We define the linear operator A_* from L_B^* into $\mathfrak{H}$ as follows:

$$A_*\Phi = a_\Phi, \quad \Phi \in L_B^*.$$

Then

$$(Ag, \Phi) = (g, A_*\Phi) = \Phi(Ag);$$
$$(\Phi, Ag) = (A_*\Phi, g) = \overline{\Phi(Ag)} \quad (g \in L,\ \Phi \in L_B^*). \tag{1.16}$$

Since

$$|(g, f)| \le \|g\|_H \|f\|_H \le \|g\|_B \|f\|_H \quad (g, f \in L),$$

we can assign to each element $f \in \mathfrak{H}$ a linear functional Φ_f so that

$$(g, \Phi_f) = (g, f) \quad (g \in L). \tag{1.17}$$

We have

$$\|\Phi_f\| = \sup_{g \in L} \frac{|\Phi_f(g)|}{\|g\|} \le \sup_{g \in L} \frac{|(g, f)|}{\|g\|} = (f, f)^{1/2} = \|f\|. \tag{1.18}$$

By virtue of (1.16) and (1.17),

$$(g, Af) = (Ag, f) = (Ag, \Phi_f) = (g, A_*\Phi_f) \quad (g \in L,\ f \in \mathfrak{H}).$$

This enables us to conclude that

$$A_*\Phi_f = \widetilde{A}f. \tag{1.19}$$

Theorem 5 *Under the conditions of Theorem* 4,

$$A_*\Phi = \sum_{j=1}^{\infty}{}_H\, \lambda_j(\Phi, \varphi_j)\,\varphi_j \tag{1.20}$$

for any $\Phi \in L_B^*$.

Proof. If we recall the proof of the inclusion $\tilde{A}\mathfrak{H} \subset L$ (see p.), we shall easily observe that (1.11) yields

$$\Phi(\widetilde{A}g) = (g, a_\Phi) = (g, A_*\Phi) \quad (g \in \mathfrak{H},\ \Phi \in L_B^*). \tag{1.21}$$

Denote by $\mathfrak{M} \subset \mathfrak{H}$ the set of all $\varphi \in \mathfrak{H}$ for which $\widetilde{A}\varphi = 0$. Taking $g = \varphi$ in (1.21), we find $A_*\Phi \in \mathfrak{N}$ where $\mathfrak{N}$ is the orthogonal complement of $\mathfrak{M}$ to $\mathfrak{H}$. On the other hand, the system of eigenvectors $\{\varphi_j\}$ is complete in $\mathfrak{N}$, so

$$A_*\Phi = \sum_{j=1}^{\infty}{}_H\, (A_*\Phi, \varphi_j)\,\varphi_j.$$

It remains to note that

$$(A_*\Phi, \varphi_j) = (\Phi, A\varphi_j) = \lambda_j(\Phi, \varphi_j) \quad (j = 1, 2, \dots).$$

Theorem 6 *Under the conditions of Theorem* 4, *for an arbitrary* $\Phi \in L_B^*$

$$A_*^2\Phi = \sum_{j=1}^{\infty}{}_B\, \lambda_j^2(\Phi, \varphi_j)\,\varphi_j. \tag{1.22}$$

Proof. Applying the operator A_* $(\widetilde{A})$ to (1.20), we get

$$A_*^2\,\Phi = \sum_{j=1}^{\infty}{}_H\, \lambda_j^2\Phi(\varphi_j)\,\varphi_j. \tag{1.23}$$

Put

$$f_n = \sum_{j=1}^{n} \lambda_j^2\Phi(\varphi_j)\,\varphi_j \in L_B \quad (n = 1, 2, \dots).$$

According to (1.15) and (1.16),

$$|\Psi(f_n - f_n)| = \Big|\sum_{j=m+1}^{n} \lambda_j^2\Phi(\varphi_j)\Psi(\varphi_j)\Big| = \Big|\sum_{j=m+1}^{n} \Phi(A\varphi_j)\Psi(A\varphi_j)\Big| \leq$$

$$\Big\{\sum_{j=m+1}^{n} |\Phi(A\varphi_j)|^2\Big\}^{1/2} p^{1/2}(\Psi) \leq \gamma\Big\{\sum_{j=m+1}^{n} |\Phi(A\varphi_j)|^2\Big\}^{1/2} \|\Psi\|$$

for any $\Psi \in L_B^*$. Choose Ψ so that $\|\Psi\| = 1$, $\Psi(f_n - f_m) = \|f_n - f_m\|_B$. Then

$$\|f_n - f_m\|_B \leq \gamma\Big\{\sum_{j=m+1}^{n} |\Phi(A\varphi_j)|^2\Big\}^{1/2}.$$

Hence the sequence $\{f_n\}$ B-converges to a certain element $f \in L$, i.e.

$$f = \sum_{j=1}^{\infty}{}_B\, \lambda_j^2\Phi(\varphi_j)\,\varphi_j.$$

The comparison of the last equality with (1.23) gives $f = A_*^2\Phi$. This completes the proof.

7. For convenience, in the sequel we assume that the space L_B is separable. In this case the unit ball $S \subset L_B^*$ ($\|\Phi\| \leq 1$) is weakly compact (see [1] or [6]) in the sense of the weak convergence of functionals. To make S a compact set we use the Schauder method [7].

Let a sequence $\{h_n\} \subset S$ be dense in S. We set

$$\|\Phi\|_1 = \sup_n \frac{|\Phi(h_n)|}{n} \quad (\Phi \in L_B^*)$$

and endow S by the distance metrics

$$d(\Phi, \Psi) = \|\Phi - \Psi\|_1 \quad (\Phi, \Psi \in S).$$

Then S becomes a compact set S_c (a metric set compact in itself). It is easy to make sure that the weak convergence of a sequence $\{\Phi_n\} \subset S$ is equivalent to its convergence in S_c-metrics. Now, we can assign to every element $f \in L$ the continuous function

$$f(\Phi) = \Phi(f) \quad (\Phi \in S)$$

so that

$$\max_{\Phi \in S} |f(\Phi)| = \|f\|_B.$$

The B-convergence of a sequence $\{f_n\} \subset L$ means the uniform convergence on S_c of the corresponding sequence of continuous functions $\{f_n(\Phi)\}$. It is not difficult to obtain the following assertion from here.

a) A sequence $\{f_n\} \subset L$ B-converges if and only if for each weakly convergent sequence of functionals $\{\Phi_n\} \subset L_B^*$, the number sequence $\{\Phi_n(f_n)\}$ is convergent.[1]

The necessity of the condition is evident. Let us prove the sufficiency. Taking the sequence $\Phi_n = \Phi$ ($n = 1, 2, \ldots; \|\Phi\| < 1$), we ascertain that the sequence of functions $\{f_n(\Phi)\}$ converges at every point of S_c. Moreover, for any sequence $\{\Phi_n\}$ convergent in S_c, the number sequence $\{f_n(\Phi_n)\}$ is convergent. This property implies the equicontinuity of the functions $f_n(\Phi)$ ($n = 1, 2, \ldots$) on S_c. By Arzelá's theorem, the sequence $\{f_n(\Phi)\}$ is a compact set with respect to the uniform convergence. Since, in addition, it converges at every point, all its limit points (in the uniform convergence) are the same, hence it is uniformly convergent. This completes the proof of a).

Definition 2 *We say that an HB-continuous operator A has a continuous kernel if*

1) *for any $\Phi \in L_B^*$,*

$$A_*\Phi \in L;$$

2) *for an arbitrary weakly convergent sequence $\{\Phi_n\} \subset L_B^*$, the sequence $\{A_*\Phi_n\} \subset L$ is B-convergent.*

[1]The proposition a) is undoubtedly valid in the case of a nonseparable L_B because every sequence belongs to a certain separable subspace.

Taking into account the proposition a), we can state that the property 2) (if 1) is assumed) is equivalent to the following:

2′) if sequences $\{\Phi_n\}, \{\Psi_n\} \subset L_B^*$ converge weakly, then the number sequence $\{(A_*\Phi_n, \Psi_n)\}$ is convergent; in other words,

$$(A_*\Phi, \Psi) \tag{1.24}$$

is a continuous function on the compact set S_c. This explains the above terminology.

We shall show that an operator A with a continuous kernel is not only H-compact but HB-compact, that is it transforms every H-bounded set into a B-compact one.

Indeed, let $\{f_n\} \subset \mathfrak{H}$, $\|f_n\| \leq 1$ $(n = 1, 2, \dots)$, and

$$\Phi_n = \Phi_{f_n} \quad (n = 1, 2, \dots).$$

By virtue of (1.18),

$$\|\Phi_n\| \leq \|f_n\| \leq 1 \quad (n = 1, 2, \dots).$$

As the ball S is weakly compact, so is the sequence $\{\Phi_n\}$. In view of 2), the sequence $\{A_*\Phi_n\}$ is B-compact. Since, by (1.19),

$$A_*\Phi_n = \widetilde{A} f_n \quad (n = 1, 2, \dots),$$

we deduce that the image of any bounded set from $\mathfrak{H}$ under the operator $\widetilde{A}$ is a B-compact set from L.

To prove Theorem 7 which generalizes Mercer's well-known theorem in the integral equation theory, following the main idea used in the proof of its classical version, we need the next lemma.

Lemma 3 *Let A be a positive*[1] *operator with a continuous kernel. Then*

$$(A_*\Phi, \Phi) \geq 0 \quad (\Phi \in L_B^*).$$

Proof. Denote by E the set of all functionals Φ_f $(f \in L)$. If all the functionals Φ_f vanish on the element $g \in L$, then $g = 0$, i.e. the set E is total. Therefore (see [1]), the weak closure of E coincides with L_B^*. Hence, for each $\Phi \in L_B^*$, there exists a sequence Φ_{f_n} $(n = 1, 2, \dots)$ weakly convergent to Φ. For this,

$$\lim_{n \to \infty} (A_*\Phi_{f_n}, \Phi_{f_n}) = (A_*\Phi, \Phi).$$

On the other hand,

$$(A_*\Phi_{f_n}, \Phi_{f_n}) = (A_*\Phi_{f_n}, f_n) = (Af_n, f_n) \geq 0 \quad (n = 1, 2, \dots).$$

This completes the proof.

[1] A linear operator A $(AL \subset L)$ is called positive if $(Af, f) \geq 0$ $(f \in L)$. Such an operator is always Hermitian.

Theorem 7 *Let a positive operator $A\,(AL \subset L)$ have a continuous kernel. Then for any $\Phi \in L_B^*$ and any $\Psi \in L_B^*$,*

$$(A_*\Phi, \Psi) = \sum_{j=1}^{\infty} \lambda_j (\Phi, \varphi_j)\overline{(\Psi, \varphi_j)}. \tag{1.25}$$

The expansion (1.25) *converges uniformly for* $\|\Phi\| \le 1,\ \|\Psi\| \le 1$.

Proof. Since the operator A is positive, $\lambda_j > 0$ $(j = 1, 2, \dots)$. Define the operators $A^{(n)}$ $(n = 1, 2, \dots)$ as follows:

$$A^{(n)}g \;=\; Ag - \sum_{j=1}^{n} \lambda_j (g, \varphi_j)\, \varphi_j \quad (n = 1, 2, \dots).$$

In view of (1.13),

$$(A^{(n)}g, g) \;=\; \sum_{j=n+1}^{\infty} \lambda_j |(g, \varphi_j)|^2 \;\ge\; 0 \quad (n = 1, 2, \dots).$$

So the operators $A^{(n)}$ $(n = 1, 2, \dots)$ are positive. Together with A, the operators $A^{(n)}$ $(n = 1, 2, \dots)$ are HB-continuous, and

$$A_*^{(n)}\Phi \;=\; A_*\Phi - \sum_{j=1}^{n} \lambda_j \Phi(\varphi_j)\, \varphi_j \quad (\Phi \in L_B^*;\ n = 1, 2, \dots).$$

It is also easily seen that like A, each operator $A^{(n)}$ $(n = 1, 2, \dots)$ satisfies the conditions 1) and 2) providing the existence of a continuous kernel for it. By Lemma 3 ,

$$(A_*^{(n)}\Phi, \Phi) \;=\; (A_*\Phi, \Phi) - \sum_{j=1}^{n} \lambda_j |(\Phi, \varphi_j)|^2 \;\ge\; 0 \quad (n = 1, 2, \dots),$$

whence

$$q(\Phi) \;=\; \Big\{ \sum_{j=1}^{\infty} \lambda_j |(\Phi, \varphi_j)|^2 \Big\}^{1/2} \;<\; \infty$$

for any $\Phi \in L_B^*$. The functional $q(\Phi)$ is continuous because of its lower semicontinuity (see Lemma 2), hence there exists a constant γ_1 such that

$$q(\Phi) \le \gamma_1 \|\Phi\| \quad (\Phi \in L_B^*).$$

The inequality

$$\Big| \Psi\Big(\sum_{j=m}^{n} \lambda_j (\Phi, \varphi_j)\, \varphi_j \Big) \Big| \le \Big\{ \sum_{j=m}^{n} \lambda_j |(\Psi, \varphi_j)|^2 \Big\}^{1/2} \Big\{ \sum_{j=m}^{n} \lambda_j |(\Phi, \varphi_j)|^2 \Big\}^{1/2} \le$$

$$\gamma_1\|\Psi\|\Big\{\sum_{j=m}^{n}\lambda_j|(\Phi,\varphi_j)|^2\Big\}^{1/2} \quad (m,n=1,2,\dots)$$

shows that the series

$$\sum_{j=1}^{\infty}\lambda_j\Phi(\varphi_j)\,\varphi_j$$

B-converges for any $\Phi\in L_B^*$. Comparing with (1.20) gives

$$A_*\Phi = \sum_{j=1}^{\infty}{}_B\,\lambda_j(\Phi,\varphi_j)\,\varphi_j \quad (\Phi\in L_B^*),$$

whence

$$(A_*\Phi,\Psi) = \sum_{j=1}^{\infty}\lambda_j(\Phi,\varphi_j)\overline{(\Psi,\varphi_j)} \quad (\Phi,\Psi\in L_B^*).$$

In particular,

$$(A_*\Phi,\Phi) = \sum_{j=1}^{\infty}\lambda_j|(\Phi,\varphi_j)|^2. \tag{1.26}$$

According to 2'), $(A_*\Phi,\Phi)$ is a continuous function on the compact set S_c. By Dini's theorem, the series (1.26) converges uniformly on S_c. Due to the Schwartz inequality, the series in the right-hand side of (1.25) is uniformly convergent with respect to $\Phi,\Psi\in S$. This completes the proof.

Remark 1 *It is not difficult to observe that the convergence of the expansion* (1.25) *uniform with respect to* $\Phi,\Psi\in S$ *is equivalent to the uniform* B*-convergence of*

$$A_*\Phi = \sum_{j=1}^{\infty}{}_B\,\lambda_j(\Phi,\varphi_j)\,\varphi_j \quad (\Phi\in L_B^*), \tag{1.27}$$

when $\|\Phi\|\le 1$. *We can also easily see that for an* HB*-continuous operator* A*, the uniform convergence of the expansion* (1.27) ($\|\Phi\|\le 1$) *is not only a necessary, but also a sufficient condition for* A *to have a continuous kernel.*

In fact, the uniform convergence of the series (1.27) ($\|\Phi\|\le 1$) means that the vector-function $A_*\Phi$ defined on S_c is a limit uniform on S_c of the sequence of functions

$$B_n(\Phi) = \sum_{j=1}^{n}\lambda_j(\Phi,\varphi_j)\,\varphi_j \quad (n=1,2,\dots)$$

continuous on S_c. Consequently, the vector-function $A_*\Phi$ is also continuous on S_c. Thus, $A_*\Phi$ transforms every weakly convergent sequence of functionals $\{f_n\}\subset S$ into the B-convergent sequence of vectors $\{A_*f_n\}$.

2. Hermitizable Operators

1. Let L be a complete Hilbert space, H a strictly positive operator defined on the whole L, that is a Hermitian operator with the property

$$(Hf, f) > 0 \quad \text{as } f \neq 0 \quad (f \in L).$$

A linear operator A on L is called H-Hermitian if the operator HA is Hermitian, i.e.

$$(HAg, f) = (Hg, Af) \quad (g, f \in L). \tag{2.1}$$

If the operator A is continuous, it has the adjoint A^*. Then the relation (2.1) means

$$HA = (HA)^* = A^*H. \tag{2.2}$$

To apply the results of the previous section to H-Hermitian operators, we consider L as a Riesz-Banach space with the norm

$$\|f\|_B = (f, f)^{1/2} \quad (f \in L),$$

in which the scalar product is introduced as

$$(g, f)_H = (Hg, f). \tag{2.3}$$

Since a Hermitian operator given on the whole L is always continuous (bounded), there exists a constant C such that

$$\|f\|_H = (Hf, f)^{1/2} \leq C \|f\|_B \quad (f \in L).$$

Thus, the condition (1.4) is fulfilled.

It is easy to observe that the norms $\|f\|_H$ and $\|f\|_B$ are not topologically equivalent if and only if the point 0 belongs to the continuous spectrum of H (this is the case, for example, when H is a compact operator). Using the B- and the H-terminologies from Section 1 and omitting B in B-terms, we can claim on the basis of Theorems 1 and 2 that every continuous (compact) H-Hermitian operator on L is H-continuous (H-compact). So, if an H-Hermitian operator A is compact, then it has an H-orthonormal fundamental system of eigenvectors $\{\varphi_i\}$:

$$A\varphi_i = \lambda_i \varphi_i \quad (i = 1, 2, \dots;\ \lambda_i \to 0).$$

Furthermore, the expansion

$$Ag = \sum_{i=1}^{\infty}{}_H \lambda_i (g, \varphi_i)_H \varphi_i = \sum_{i=1}^{\infty}{}_H \lambda_i (Hg, \varphi_i) \varphi_i \quad (g \in L) \tag{2.4}$$

holds. The H-orthonormality of φ_i means

$$(H\varphi_i, \varphi_k) = \delta_{ik} \quad (i, k = 1, 2, \dots). \tag{2.5}$$

Put

$$\psi_i = H\varphi_i \quad (i = 1, 2, \dots). \tag{2.6}$$

Then (2.4) can be written in the form

$$Ag = \sum_{i=1}^{\infty}{}_H \lambda_i (g, \psi_i)\, \varphi_i. \tag{2.7}$$

In view of (2.2),

$$A^*\psi_i = A^*H\varphi_i = HA\varphi_i = \lambda_i H\varphi_i = \lambda_i \psi_i \quad (i = 1, 2, \dots).$$

Taking into account that for the compact operator A, the number of linearly independent solutions of the equations $A\varphi - \lambda\varphi = 0$ and $A^*\psi - \lambda\psi = 0$ $(\lambda \neq 0)$ is the same, we infer that $\{\psi_i\}$ is a fundamental system of eigenvectors for the operator A^*. It follows from (2.5) and (2.6) that the systems $\{\varphi_i\}$ and $\{\psi_i\}$ are biorthogonal:

$$(\varphi_i, \psi_k) = \delta_{ik} \quad (i, k = 1, 2, \dots).$$

2. There is only one strictly positive operator $H^{1/2}$ whose square equals H. As

$$\|f\|_H^2 = (Hf, f) = \|H^{1/2} f\|^2 \quad (f \in L),$$

the equality

$$f = \sum_{k=1}^{\infty}{}_H f_k \quad (f \in L,\ f_k \in L,\ k = 1, 2, \dots)$$

is nothing but

$$H^{1/2} f = \sum_{k=1}^{\infty} H^{1/2} f_k.$$

Therefore, (2.7) is equivalent to

$$H^{1/2} Ag = \sum_{i=1}^{\infty} \lambda_i (g, \psi_i) H^{1/2} \varphi_i.$$

Applying the operator $H^{1/2}$ to both sides of the last equality, we get

$$A^*Hg = HAg = \sum_{i=1}^{\infty} \lambda_i (g, \psi_i)\, \psi_i.$$

The substitution $Hg = f$ yields

$$A^* f = \sum_{i=1}^{\infty} \lambda_i (f, \varphi_i)\, \psi_i \quad (f \in \mathcal{R}(H^{1/2})) \tag{2.8}$$

($\mathcal{R}(H^{1/2})$ denotes the range of the operator $H^{1/2}$).

3. Now we shall find the necessary and sufficient conditions for a linear operator A $(AL \subset L)$ to be HB-continuous. Since a continuous linear functional $\Phi \in L^*$ can be represented in the form

$$\Phi(g) = (g, f_\Phi) \quad (g \in L)$$

where f_Φ is a certain element from L, a continuous linear operator A $(AL \subset L)$ is HB-continuous if for any $f \in L$, the functional (Ag, f) $(g \in L)$ is continuous on L_H. The latter is equivalent to the H-boundedness of this functional, that is to the existence of a constant $C = C_f$ such that

$$|(g, A^*f)|^2 = |(Ag, f)|^2 \leq C \|g\|_H^2 = C(Hg, g) \quad (g \in L). \tag{2.9}$$

On the other hand, for $e \in L$, the inequality

$$\gamma = \sup_{g \in L} \frac{|(g, e)|}{(Hg, g)} < \infty$$

is valid if and only if $e \in \mathcal{R}(H^{1/2})$. Indeed, the substitution $g = H^{-1/2}f$ in the relation

$$|(g, e)|^2 \leq \gamma (Hg, g) = \gamma \|H^{1/2}g\|^2$$

yields $|(H^{-1/2}f, e)| \leq \gamma^{1/2}\|f\|$ $(f \in \mathcal{D}(H^{-1/2}),\ \mathcal{D}(H^{-1/2}) = \mathcal{R}(H^{1/2})$ is the domain of $H^{-1/2})$. So $(H^{-1/2}f, e)$ is a continuous linear functional on the set $\mathcal{R}(H^{1/2})$ dense in L. Hence there exists a vector $a \in L$ such that

$$(H^{-1/2}f, e) = (f, a) \quad (f \in \mathcal{D}(H^{-1/2})).$$

Taking into account the self-adjointness of the operator $H^{-1/2} = (H^{1/2})^{-1}$, we obtain

$$e \in \mathcal{D}(H^{-1/2}) = \mathcal{R}(H^{1/2}), \quad a = H^{-1/2}e, \quad e = H^{1/2}a.$$

Conversely, if $e = H^{1/2}a$, then

$$|(g, e)|^2 = |(H^{1/2}g, a)|^2 \leq \|a\|^2 \|H^{1/2}g\|^2 = \|a\|^2 (Hg, g) \quad (g \in L),$$

whence $\gamma = \|a\|^2 < \infty$. Thus, the inequality (2.9) where $C = C_f > 0$ is possible if and only if $e = A^*f \in \mathcal{R}(H^{1/2})$.

We have proved the following assertion.

1°. *In order that a continuous*[1] *linear operator A $(AL \subset L)$ be HB-continuous, it is necessary and sufficient that $\mathcal{R}(A^*) \subset \mathcal{R}(H^{1/2})$.*

Now, it is not hard to establish the next theorem.

Theorem 8 *Let an H-Hermitian operator A be compact and $\mathcal{R}(A^*) \subset \mathcal{R}(H^{1/2})$. Then the operator A has an H-orthonormal system of eigenvectors $\{\varphi_i\}$ such that*

$$Ag = \sum_{i=1}^{\infty} \lambda_i (g, \psi_i) \varphi_i \quad (g \in L), \tag{2.10}$$

[1]Recall that only a continuous operator may be HB-continuous.

$$A^*g = \sum_{i=1}^{\infty} \lambda_i(g, \varphi_i)\psi_i \quad (g \in L), \tag{2.11}$$

where $\psi_i = H\varphi_i$ $(i = 1, 2, \dots)$, *hence*

$$(\psi_i, \varphi_k) = \delta_{ik} \quad (i, k = 1, 2, \dots).$$

Proof. Under the conditions of the theorem, the H-Hermitian operator A is compact and HB-continuous. So we may apply Theorem 3 to A . This leads to the expansion (2.10). Let

$$A_n g = \sum_{i=1}^{n} \lambda_i(g, \psi_i)\varphi_i \quad (n = 1, 2, \dots).$$

Since for any $g \in L$, $A_n g \to Ag$ $(n \to \infty)$, there exists a constant $C > 0$ such that

$$\|A_n^*\| = \|A_n\| \le C. \tag{2.12}$$

Obviously,

$$A_n^* g = \sum_{i=1}^{n} \lambda_i(g, \varphi_i)\psi_i \quad (n = 1, 2, \dots).$$

By (2.8), the sequence $\{A_n^* g\}$ converges to A^*g on the set $\mathcal{R}(H^{1/2})$ dense in L. Moreover, due to (2.12)

$$\|A_n^* g - A_n^* f\| \le \|A_n^*\| \cdot \|g - f\| \le C\,\|g - f\| \quad (n = 1, 2, \dots),$$

i.e. the operators A_n^* are equicontinuous on L. Consequently, for any $g \in L$, the sequence $\{A_n^* g\}$ has a limit that depends continuously on g, and hence coincides with A^*g. The proof is complete.

4. It is evident that the conditions of Theorem 8 are fulfilled if

$$A = SH,$$

where S is a Hermitian operator and at least one of the operators S or H is compact. We have that very case if an HB-continuous operator A has a continuous kernel.

Indeed, according to Definition 2, the first condition for an HB-continuous operator A to have a continuous kernel lies in the existence for each functional $\Phi_f(g) = (g, f)$ continuous on L of an element $A_*\Phi_f \in L$ such that

$$\Phi_f(Ag) = (g, A_*\Phi_f)_H,$$

or

$$(Ag, f) = (Hg, A_*\Phi_f), \tag{2.13}$$

which is the same. Setting

$$S_1 f = A_*\Phi_f \quad (f \in L),$$

we obtain from (2.13)

$$A^* = HS_1. \tag{2.14}$$

Recalling that the second condition of Definition 2 requires the operator A_* to map any weakly convergent sequence of functionals $\{\Phi_n\} \subset L^*$ into the convergent one $\{A_*\Phi_n\} \subset L$, which means the compactness of S_1, we arrive at the following conclusion.

2°. *A linear operator A $(AL \subset L)$ has a continuous kernel if and only if it can be represented in the form*

$$A = SH \tag{2.15}$$

where the operator S is compact.

In accordance with (2.14), $S = S_1^*$, $S_1 = S^*$. We have also used the fact that the adjoint of a compact operator is compact, too.

If, in addition, the operator A is H-Hermitian, i.e. $HA = A^*H$, then we get the condition on S

$$HSH = HS^*H, \quad (SHg, Hf) = (S^*Hg, Hf) \quad (g, f \in L).$$

In view of the density of $\mathcal{R}(H)$ in L, this condition is equivalent to $S = S^*$.

Analogously, if the operator A is H-positive, i.e.

$$(HAf, f) \geq 0 \quad (f \in L),$$

then the operator S is positive:

$$(Sf, f) \geq 0 \quad (f \in L).$$

It is not difficult to obtain the next statement from what has been said.

Theorem 9 *Let H $(HL \subset L)$ be a strictly positive operator, and S $(SL \subset L)$ a compact positive one. Then there exist an H-orthogonal system of vectors $\{\varphi_i\}$ and a sequence of positive numbers $\{\lambda_i\}$ approaching zero, such that*

$$Sf = \sum_{i=1}^{\infty} \lambda_i (f, \varphi_i)\varphi_i \quad (f \in L). \tag{2.16}$$

The expansion (2.16) *converges uniformly on the unit ball $\|f\| \leq 1$.*

Proof. Let $A = SH$. It follows from the above discussion that the HB-continuous operator A has a continuous kernel. Moreover, for an arbitrary element $f \in L$ and the corresponding functional Φ_f $(\Phi_f(g) = (g, f),\ g \in L)$,

$$A_*\Phi_f = Sf, \quad (\varphi_i, \Phi_f) = (\varphi_i, f) \quad (i = 1, 2, \dots).$$

Thus, the expansion (2.16) is none other than an interpretation of (1.27) concerning the case under consideration. As far as the expansion (1.27) converges uniformly on the ball $\|\Phi\| \leq 1$, so does the series (1.16) for $\|f\| = \|\Phi_f\| \leq 1$.

Conclusion. *If a linear operator A $(AL \subset L)$ is of the form $A = SH$ where S is a positive compact operator, then the expansions* (2.10) *and* (2.11) *converge uniformly on the unit ball $\|g\| \leq 1$.*

In fact, the expansion (2.10) can be derived from the equality (2.16) if we set $f = Hg$ there; we may obtain (2.11) by applying the continuous operator H to both sides of (2.16), whence the desired assertion follows.

Remark 2 . *It is interesting to note that Theorem* 9 *is an analogue of Mercer's theorem from the integral equation theory, for both theorems are different specific cases of Theorem* 7.

Remark 3 *If the operator S is strictly positive ($Sf \neq 0$ as $f \neq 0$), then, by virtue of* (2.16), *there is no vector in L orthogonal to all vectors φ_i ($i = 1, 2, \dots$). So the span of the system $\{\varphi_i\}$ is dense in L, hence in L_H. As the system $\{\varphi_i\}$ is H-orthonormal,*

$$(g,g)_H = \sum_{i=1}^{\infty} |(g,\varphi_i)_H|^2, \text{ i.e. } (Hg,g) = \sum_{i=1}^{\infty} |(Hg,\varphi_i)|^2 \quad (g \in L).$$

Putting $Hg = f$, $g = H^{-1}f$ here , we get

$$(H^{-1}f,f) = \sum_{i=1}^{\infty} |(f,\varphi_i)|^2 \quad (f \in \mathcal{D}(H^{-1}) = \mathcal{R}(H)).$$

On the other hand, in view of (2.16),

$$(Sf,f) = \sum_{i=1}^{\infty} \lambda_i |(f,\varphi_i)|^2 \quad (f \in L).$$

As we can observe, by using the system $\{\varphi_i\}$, the forms $(H^{-1}f,f)$ and (Sf,f) are simultaneously reduced to the sum of squares.[1]

3. Asymptotics of the Characteristic Numbers of a Compact Hermitian Operator

As before, L denotes a complete Hilbert space and H ($HL \subset L$) a strictly positive compact Hermitian operator. We shall be interested in compact operators of the form

$$A = H(I + R) \tag{3.1}$$

where R is a compact operator.

If λ is not a characteristic number of A, then the Fredholm resolvent A_λ determined by the equation

$$(I - \lambda A)^{-1} = I + \lambda A_\lambda$$

exists, and

$$A_\lambda = A + \lambda A_\lambda A = A + \lambda A A_\lambda. \tag{3.2}$$

In particular, it follows from (3.2) that the operator A_λ is compact. The substitution of the expression (3.1) for A in (3.2) gives

$$A_\lambda = H(I + R^{(\lambda)}),$$

[1]This is also true in the case where the set L_0 of all solutions f of the equation $Sf = 0$ is different from $\{0\}$. We need only add an H-orthonormal basis of L_0 to the system $\{\varphi_i\}$.

where the operator

$$R^{(\lambda)} = R + \lambda A_\lambda + \lambda R A_\lambda$$

is also compact.

It should be noted that if μ is an r-multiplicity characteristic number (that is $\mu \neq 0$ and μ^{-1} is an eigenvalue of the operator A of the multiplicity r. In this section, it is convenient for us to deal with characteristic number) of the operator A and $\varphi = \mu A\varphi$ $(\varphi \neq 0)$, then, by (3.2), $\varphi = (\mu - \lambda)A_\lambda\varphi$. So, $\mu - \lambda$ is an r-multiplicity characteristic number of the operator A_λ.

We shall prove below that an operator A of the form (3.1) may have at most a finite number of negative characteristic numbers. It follows from here that the characteristic numbers $\mu_k(A)$ $(k = 1, 2, \dots)$ of the operator A can be enumerated so that

$$\mu_1(A) \leq \mu_2(A) \leq \mu_3(A) \leq \dots \quad (\mu_n(A) \to \infty).$$

In the sequence $\{\mu_n(A)\}$, every characteristic number is taken as many times as its multiplicity. In the same way we enumerate the characteristic numbers $\mu_k(H)$ $(k = 1, 2, \dots)$ of the operator H:

$$\mu_1(H) \leq \mu_2(H) \leq \mu_3(H) \leq \dots.$$

Theorem 10 *Let H $(HL \subset L)$ be a strictly positive compact operator, and A a Hermitian operator of the form*

$$A = H(I + R)$$

where R is a compact operator. Then the operator A has at most a finite number of negative characteristic numbers and

$$\lim_{n\to\infty} \frac{\mu_n(A)}{\mu_n(H)} = 1. \tag{3.3}$$

Proof. Let $\{\varphi_k\}$ be a fundamental orthonormal system of eigenvectors of the operator H, corresponding to the sequence $\{\mu_k(H)\}$, i.e.

$$\varphi_j = \mu_j(H)H\varphi_j, \quad (\varphi_j, \varphi_k) = \delta_{jk} \quad (j, k = 1, 2, \dots).$$

In as far as the operator H is strictly positive, the orthonormal system $\{\varphi_k\}$ is complete in L.

Denote by L_n $(n = 1, 2, \dots)$ the set of all $g \in L$ such that

$$(g, \varphi_1) = (g, \varphi_2) = \dots = (g, \varphi_n) = 0. \tag{3.4}$$

It is known that

$$\frac{1}{\mu_n(H)} = \max_{\varphi \in L_{n-1}} \frac{(H\varphi, \varphi)}{(\varphi, \varphi)} \quad (n = 1, 2, \dots). \tag{3.5}$$

The maximum is reached at $\varphi = \varphi_n$. As the operator R is H-Hermitian and compact, it is H-compact. So the form $(HRf, f) = (Rf, f)_H$ is H-weakly continuous on the H-ball $S_H = \{g : (g, g)_H = (Hg, g) \leq 1\}$. Since the equations (3.4) can be written in the form

$$(g, \varphi_1)_H = (g, \varphi_2)_H = \dots = (g, \varphi_n)_H = 0,$$

any sequence $\{g_n\}$ with the properties

$$g_n \in S_H, \quad g_n \in L_n \quad (n = 1, 2, \dots)$$

converges H-weakly to zero, hence

$$\lim_{n\to\infty} (Rg_n, g_n)_H = 0.$$

It follows from here that

$$E_n = \sup_{g\in L_n} \frac{(HRg, g)}{(Hg, g)} \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

Choose a natural number ν so that

$$(HRg, g) < (Hg, g) \quad (g \in L_\nu).$$

By virtue of (3.1),

$$(Ag, g) = (Hg, g) + (HRg, g) > 0 \quad (g \in L_\nu).$$

This shows that the operator A has at most ν negative characteristic numbers (taking into account their multiplicities). In the opposite case, we should find an orthonormal system of eigenfunctions $\{\psi_1, \dots, \psi_{\nu+1}\}$ such that

$$\psi_j = \mu_j A\psi_j, \quad \mu_j < 0 \quad (j = 1, 2, \dots, \nu + 1).$$

But then we could find a vector $g \in L_\nu$, $g = c_1\psi_1 + \cdots + c_{\nu+1}\psi_{\nu+1} \neq 0$, for which

$$(Ag, g) = \sum_{j=1}^{\nu+1} \mu_j^{-1} |c_j|^2 < 0.$$

Now, without loss of generality, we may assume that the operator A is strictly positive. Otherwise, we could take the appropriate positive operator A_λ; moreover ,

$$\lim_{n\to\infty} \frac{\mu_n(A_\lambda)}{\mu_n(A)} = \lim_{n\to\infty} \frac{\mu_n(A) - \lambda}{\mu_n(A)} = 1.$$

The minimax property of the characteristic numbers $\mu_n(A)$ $(n = 1, 2, \dots)$ (see [8]) leads to the inequality

$$\frac{1}{\mu_n(A)} \leq \sup_{g\in L_{n-1}} \frac{(Ag, g)}{(g, g)} \quad (n = 1, 2, \dots).$$

On the other hand, by (3.5) and (3.6),

$$\sup_{g\in L_{n-1}} \frac{(Ag, g)}{(g, g)} = \sup_{g\in L_{n-1}} \left(\frac{(Hg, g)}{(g, g)} \left(1 + \frac{(HRg, g)}{(Hg, g)} \right) \right) \leq$$

$$\left(1 + \sup_{g\in L_{n-1}} \frac{(HRg, g)}{(Hg, g)} \right) \sup_{g\in L_{n-1}} \frac{(Hg, g)}{(g, g)} = (1 + E_{n-1}) \cdot \frac{1}{\mu_n(H)} \quad (n = 1, 2, \dots).$$

Thus,

$$\limsup_{n\to\infty} \frac{\mu_n(H)}{\mu_n(A)} \leq 1.$$

As the operator A is strictly positive, we can interchange the operators A and B because (3.1) implies

$$H = A(I+R)^{-1} = A(I-R_{-1}),$$

where R_λ is the resolvent of the operator R. Consequently,

$$\limsup_{n\to\infty} \frac{\mu_n(A)}{\mu_n(H)} \leq 1,$$

which proves (3.3).

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Note

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