Acta Math. Hungar.
109 (1–2) (2005), 1–14.

ON THE NONNEGATIVITY OF OPERATOR PRODUCTS

S. HASSI (Vaasa), Z. SEBESTYÉN (Budapest) and H. S. V. DE SNOO (Groningen)*

Abstract. A bounded, not necessarily everywhere defined, nonnegative operator A in a Hilbert space $\mathfrak{H}$ is assumed to intertwine in a certain sense two bounded everywhere defined operators B and C. If the range of A is provided with a natural inner product then the operators B and C induce two new operators on the completion space. This construction is used to show the existence of selfadjoint and nonnegative extensions of B^*A and C^*A.

1. Introduction

In 1963 E. P. Wigner [20] introduced the notion of weakly positive matrices in order to give a generalization of a theorem of F. Riesz about the positivity of the product of two commuting positive Hilbert space operators. The more interesting result of Wigner says that any selfadjoint product of at most three positive operators is in fact positive. Moreover, he gave an example of a selfadjoint product of four positive definite 2×2 matrices which is not positive. Wigner's result was generalized by S. J. Bernau [3], who showed that the selfadjoint product of a positive operator and an operator with positive spectrum is positive.

The basic setting in the present paper is the case of a nonnegative operator $A \in [\mathfrak{H}]$ and a pair of operators B and C in $[\mathfrak{H}]$ which satisfy $C^*A = AB$. Here $\mathfrak{H}$ is a Hilbert space and $[\mathfrak{H}]$ denotes the set of all bounded everywhere defined linear operators in $\mathfrak{H}$. In fact, a slightly more general operator A will be considered, namely a nonnegative linear operator A acting in $\mathfrak{H}$ whose domain of definition, denoted by $\operatorname{dom} A$, is not necessarily dense in $\mathfrak{H}$ or closed. It will be assumed that this operator has nonnegative extensions in $[\mathfrak{H}]$. The

*The authors gratefully acknowledge the support by a bilateral agreement between the Eötvös Loránd University in Budapest and the University of Groningen, and by the Research Institute for Technology of the University of Vaasa.

Key words and phrases: nonnegative operator, symmetric operator, selfadjoint operator, product of operators, Friedrichs extension, Kreĭn–von Neumann extension.

2000 Mathematics Subject Classification: primary 47A05, 47B15, 47B25; secondary 47A10, 47B65.

0236–5294/$ 20.00 © 2005 Akadémiai Kiadó, Budapest

range of the operator A, denoted by $\operatorname{ran} A$, will be provided with a suitable inner product, and will then be completed to a Hilbert space $\mathfrak{H}_A$. The pair of operators B and C in $[\mathfrak{H}]$ is now required to satisfy

$$\begin{cases} B(\operatorname{dom} A) \subset \operatorname{dom} A, & C^*A = AB \restriction \operatorname{dom} A, \\ C(\operatorname{dom} A) \subset \operatorname{dom} A, & B^*A = AC \restriction \operatorname{dom} A. \end{cases} \tag{1.1}$$

Then the pair B and C in $[\mathfrak{H}]$ induces a pair of associated operators $\widehat{B}$ and $\widehat{C}$ in $[\mathfrak{H}_A]$.

As a consequence of these constructions a number of criteria are obtained for the existence of selfadjoint and nonnegative extensions in $[\mathfrak{H}]$ of C^*A and B^*A. One typical result concerns the situation of a nonnegative operator $A \in [\mathfrak{H}]$ and an operator $B \in [\mathfrak{H}]$ such that the product AB is selfadjoint, cf. [17]. The spectrum $\sigma(\widehat{B})$ of $\widehat{B}$ is, in fact, a subset of the spectrum $\sigma(B)$ of B, and it is real due to the selfadjointness of $\widehat{B}$. As a consequence the product AB is nonnegative if $\sigma(B)$ does not contain negative real numbers. Another approach (via unbounded operators) to such results is given in [19].

Throughout this paper the spectral radius of an operator T is denoted by $r(T)$ and its spectrum and resolvent set are denoted by $\sigma(T)$ and $\rho(T)$, respectively.

2. Hilbert space completions and associated operators

Let A be a nonnegative linear operator in the Hilbert space $\mathfrak{H}$ whose domain of definition is not necessarily dense in $\mathfrak{H}$ or closed. Let $\widetilde{A}$ be a nonnegative operator in $[\mathfrak{H}]$ which extends the operator A. The operator Schwarz inequality

$$\|\widetilde{A}x\|^2 \leqq \|\widetilde{A}\|\,(\widetilde{A}x, x), \quad x \in \mathfrak{H},$$

implies an operator Schwarz inequality for A:

$$\|Ax\|^2 \leqq m(Ax, x), \quad x \in \operatorname{dom} A, \tag{2.1}$$

with some constant $m \geqq 0$.

Now assume that A is a nontrivial linear operator in the Hilbert space $\mathfrak{H}$ whose domain of definition is not necessarily dense in $\mathfrak{H}$ or closed, such that the inequality (2.1) holds for some $m > 0$. Equip the range space $\operatorname{ran} A$ with the inner product

$$\langle Ax, Ay\rangle = (Ax, y), \quad x, y \in \operatorname{dom} A, \tag{2.2}$$

and let $\mathfrak{H}_A$ denote its Hilbert space completion. Let $J : \operatorname{ran} A \subset \mathfrak{H}_A \to \operatorname{ran} A \subset \mathfrak{H}$ be the natural embedding of $\mathfrak{H}_A$ into $\mathfrak{H}$, defined by the identification

$$J(Ax) = Ax, \quad x \in \operatorname{dom} A. \tag{2.3}$$

It follows from (2.1) that the operator J is continuous with norm bound $\sqrt{m}$. Hence the closure of J has the same bound. It follows from

$$\langle J^*x, Ay\rangle = \big(x, J(Ay)\big) = (x, Ay) = \langle Ax, Ay\rangle, \quad x, y \in \operatorname{dom} A,$$

that the adjoint J^* from $\mathfrak{H}$ to $\mathfrak{H}_A$ satisfies the characteristic identity

$$J^*x = Ax, \quad x \in \operatorname{dom} A. \tag{2.4}$$

The double adjoint J^{**} is the closure of J, so that it is a bounded linear operator from $\mathfrak{H}_A$ to $\mathfrak{H}$ with norm bound $\sqrt{m}$. Since (2.4) implies that the range $\operatorname{ran} J^*$ is dense in $\mathfrak{H}_A$, it follows that the mapping J^{**} is one-to-one on $\mathfrak{H}_A$. The operator $J^{**}J^*$ is selfadjoint and nonnegative in $[\mathfrak{H}]$. Since J^{**} is the closure of J, the identities (2.3) and (2.4) imply that $J^{**}J^*$ is an extension of the operator A. In fact, $J^{**}J^*$ is the Kreĭn–von Neumann extension, the smallest nonnegative extension in $[\mathfrak{H}]$ among all nonnegative selfadjoint extensions of A.

THEOREM 2.1. *The completion $\mathfrak{H}_A$ of* $\operatorname{ran} A$ *induced by the inner product* (2.2) *equals* $\operatorname{ran} J^{**}$. *In particular, when $A \in [\mathfrak{H}]$, then* $\operatorname{ran} J^{**} = \operatorname{ran} A^{\frac{1}{2}}$.

The identification of $J^{**}J^*$ as the Kreĭn–von Neumann extension of the operator A can be found in [18], see also [16]. The statement in Theorem 2.1 can be found in [2], [13]. A parametrization of all nonnegative selfadjoint extensions of the operator A can be found in [7]. Note that the condition (2.1) is necessary and sufficient for the nonnegative operator A to have a bounded Kreĭn–von Neumann extension, cf. [1], [8].

Let B and C be operators in $[\mathfrak{H}]$ which intertwine the nonnegative bounded operator A in the sense of (1.1). Introduce operators $\widehat{B}$ and $\widehat{C}$ from $\operatorname{ran} A \subset \mathfrak{H}_A$ to $\operatorname{ran} A \subset \mathfrak{H}_A$ via the following definitions:

$$\widehat{B}(Ax) = ABx, \quad \widehat{C}(Ax) = ACx, \quad x \in \operatorname{dom} A. \tag{2.5}$$

It follows from (1.1) that these operators are well defined.

THEOREM 2.2. *The operators $\widehat{B}$ and $\widehat{C}$ have unique extensions in $[\mathfrak{H}_A]$, again denoted by $\widehat{B}$ and $\widehat{C}$, which satisfy*

$$\|\widehat{B}\|_A = \|\widehat{C}\|_A \leqq r(CB)^{\frac{1}{2}} = r(BC)^{\frac{1}{2}}. \tag{2.6}$$

Moreover, these operators satisfy the duality

$$\widehat{B}^* = \widehat{C}. \tag{2.7}$$

PROOF. A repeated application of the conditions in (1.1) leads to the identities

(2.8)
$$(ACBx,y)=(Ax,CBy),\quad (ABCx,y)=(Ax,BCy),\quad x\in\operatorname{dom}A,\quad y\in\mathfrak{H}.$$

The definitions (2.5) of $\widehat{B}$ and $\widehat{C}$, the definition (2.2), and the conditions in (1.1) yield:

$$\begin{aligned}(2.9)\qquad \big\langle \widehat{B}(Ax),\widehat{B}(Ax)\big\rangle &= \langle ABx,ABx\rangle=(ABx,Bx)\\ &=(Ax,CBx)=\langle Ax,ACBx\rangle,\quad x\in\operatorname{dom}A,\end{aligned}$$

and

$$\begin{aligned}(2.10)\qquad \big\langle \widehat{C}(Ax),\widehat{C}(Ax)\big\rangle &= \langle ACx,ACx\rangle=(ACx,Cx)\\ &=(Ax,BCx)=\langle Ax,ABCx\rangle,\quad x\in\operatorname{dom}A,\end{aligned}$$

It follows from (2.9), the Schwarz inequality for semi-inner products, the definition (2.2), and (2.8) that

$$\big\langle \widehat{B}(Ax),\widehat{B}(Ax)\big\rangle \leqq \langle Ax,Ax\rangle^{\frac12}\langle ACBx,ACBx\rangle^{\frac12}$$

$$=\langle Ax,Ax\rangle^{\frac12}(ACBx,CBx)^{\frac12}=\langle Ax,Ax\rangle^{\frac12}\big(Ax,(CB)^2x\big)^{\frac12},\quad x\in\operatorname{dom}A,$$

which results for each $n\in\mathbf{N}$ in

$$(2.11)\quad \big\langle \widehat{B}(Ax),\widehat{B}(Ax)\big\rangle \leqq \langle Ax,Ax\rangle^{1-2^{-n}}\big(Ax,(CB)^{2^n}x\big)^{2^{-n}},\quad x\in\operatorname{dom}A,$$

due to the induction step

$$\begin{aligned}\big(Ax,(CB)^{2^n}x\big)&=\big\langle Ax,A(CB)^{2^n}x\big\rangle \leqq \langle Ax,Ax\rangle^{\frac12}\big\langle A(CB)^{2^n}x,A(CB)^{2^n}x\big\rangle^{\frac12}\\ &=\langle Ax,Ax\rangle^{\frac12}\big(A(CB)^{2^n}x,(CB)^{2^n}x\big)^{\frac12}\\ &=\langle Ax,Ax\rangle^{\frac12}\big(Ax,(CB)^{2^{n+1}}x\big)^{\frac12},\quad x\in\operatorname{dom}A.\end{aligned}$$

Similarly, it follows from (2.10) that for each $n\in\mathbf{N}$

$$(2.12)\quad \big\langle \widehat{C}(Ax),\widehat{C}(Ax)\big\rangle \leqq \langle Ax,Ax\rangle^{1-2^{-n}}\big(Ax,(BC)^{2^n}x\big)^{2^{-n}},\quad x\in\operatorname{dom}A.$$

Hence, from (2.11) and (2.12) one obtains for the case $Ax \neq 0$ the following inequalities

$$\left\langle \widehat{B}(Ax), \widehat{B}(Ax) \right\rangle \leqq \langle Ax, Ax \rangle \left[\frac{\|Ax\| \, \|x\|}{\langle Ax, Ax \rangle} \right]^{2^{-n}} \left\| (CB)^{2^n} \right\|^{2^{-n}}, \quad x \in \operatorname{dom} A,$$

and

$$\left\langle \widehat{C}(Ax), \widehat{C}(Ax) \right\rangle \leqq \langle Ax, Ax \rangle \left[\frac{\|Ax\| \, \|x\|}{\langle Ax, Ax \rangle} \right]^{2^{-n}} \left\| (BC)^{2^n} \right\|^{2^{-n}}, \quad x \in \operatorname{dom} A.$$

In the limit, as $n \to \infty$, these inequalities lead to

$$\left\langle \widehat{B}(Ax), \widehat{B}(Ax) \right\rangle \leqq r(CB) \langle Ax, Ax \rangle, \quad x \in \operatorname{dom} A, \tag{2.13}$$

and to

$$\left\langle \widehat{C}(Ax), \widehat{C}(Ax) \right\rangle \leqq r(BC) \langle Ax, Ax \rangle, \quad x \in \operatorname{dom} A. \tag{2.14}$$

Observe that $r(CB) = r(BC)$ since the nonzero points in the resolvent sets $\rho(CB)$ and $\rho(BC)$ of the operators CB and BC in $[\mathfrak{H}]$ are the same. The estimates (2.13) and (2.14) show that both operators $\widehat{B}$ and $\widehat{C}$ can be extended to operators in $[\mathfrak{H}_A]$ and are both bounded in norm by $r(CB)^{\frac{1}{2}} = r(BC)^{\frac{1}{2}}$.

It follows from the definitions in (2.5), the definition (2.2), and the conditions (1.1) that the following intertwining relation is satisfied:

$$\begin{aligned} \left\langle \widehat{B}(Ax), Ay \right\rangle &= \langle ABx, Ay \rangle = (ABx, y) = (C^* Ax, y) \\ &= (Ax, Cy) = \langle Ax, ACy \rangle = \left\langle Ax, \widehat{C}(Ay) \right\rangle, \quad x, y \in \operatorname{dom} A. \end{aligned} \tag{2.15}$$

Since both operators $\widehat{B}$ and $\widehat{C}$ belong to $[\mathfrak{H}_A]$, the identity (2.15) leads to the identity (2.7) and, hence, to the equality $\|\widehat{B}\|_A = \|\widehat{C}\|_A$ in (2.6). $\square$

The extensions $\widehat{B}$ and $\widehat{C}$ in $[\mathfrak{H}_A]$ of the operators defined in (2.5) will be called the *associated operators* (with respect to the operators B and C in $[\mathfrak{H}]$). The associated operators in this form first appear in [18]. The construction of the new Hilbert space and the associated operators goes back to J. A. Dieudonné [4], M. G. Kreĭn [9], P. D. Lax [10], W. T. Reid [14], A. C. Zaanen [22], and H. J. Zimmerberg [24].

COROLLARY 2.3. *The operators $J^{**}\widehat{B}J^*$ and $J^{**}\widehat{C}J^*$ in $[\mathfrak{H}]$ form a dual pair of extensions in the following sense:*

(2.16)
$$C^*A \subset J^{**}\widehat{B}J^* \subset A^*B = (B^*A)^*, \quad B^*A \subset J^{**}\widehat{C}J^* \subset A^*C = (C^*A)^*,$$

where A^*B and A^*C, and the adjoints of B^*A and C^*A, are understood in the sense of linear relations. The relations A^*B and A^*C are operators precisely when $\operatorname{dom} A$ *is dense in $\mathfrak{H}$.*

PROOF. Clearly, the operators $J^{**}\widehat{B}J^*$ and $J^{**}\widehat{C}J^*$ belong to $[\mathfrak{H}]$. Since J^{**} is the closure of J, the definitions in (2.3), (2.4), and (2.5), and the conditions (1.1) lead to

$$J^{**}\widehat{B}J^*x = ABx, \quad J^{**}\widehat{C}J^*x = ACx, \quad x \in \operatorname{dom} A. \tag{2.17}$$

In other words, $J^{**}\widehat{B}J^*$ and $J^{**}\widehat{C}J^*$ extend $AB \restriction \operatorname{dom} A$ and $AC \restriction \operatorname{dom} A$, respectively. By (1.1) this is equivalent to

$$C^*A \subset J^{**}\widehat{B}J^*, \quad B^*A \subset J^{**}\widehat{C}J^*.$$

Due to (2.7), the dual relations in (2.16) follow by taking adjoints. □

Corollary 2.3 depends on some results concerning the product of linear relations (i.e., multivalued linear operators). For completeness these results are presented here with proofs.

LEMMA 2.4. *Let S and T be linear relations in a Hilbert space $\mathfrak{H}$. Then*

$$(ST)^* \supset T^*S^*. \tag{2.18}$$

Moreover, if $S \in [\mathfrak{H}]$, then

$$(ST)^* = T^*S^*. \tag{2.19}$$

In this case T^*S^* is an operator if and only if the linear relation T is densely defined.

PROOF. If $\{f, g\} \in T^*S^*$, then there exists an element $\varphi \in \mathfrak{H}$ such that $\{f, \varphi\} \in S^*$ and $\{\varphi, g\} \in T^*$. Likewise, if $\{h, k\} \in ST$, then there exists an element $\psi \in \mathfrak{H}$ such that $\{h, \psi\} \in T$ and $\{\psi, k\} \in S$. The duality between S and S^*, and between T and T^*, leads to the following identities:

$$(\varphi, \psi) = (f, k), \quad (g, h) = (\varphi, \psi).$$

Therefore, it follows that

$$(g,h)-(f,k)=(\varphi,\psi)-(\varphi,\psi)=0,$$

so that $\{f,g\}\in(ST)^*$. This shows the inclusion in (2.18).

Assume that $S\in[\mathfrak{H}]$. To show (2.19) it suffices to show the reverse inclusion in (2.18). Let $\{f,g\}\in(ST)^*$, so that $(g,h)=(f,k)$ for all $\{h,k\}\in ST$. Each element $\{h,k\}\in ST$ is of the form $\{h,\psi\}\in T$ and $\{\psi,S\psi\}\in S$, and conversely, if $\{h,\psi\}\in T$ and $\{\psi,S\psi\}\in S$, then $\{h,S\psi\}\in ST$. Therefore

$$0=(g,h)-(f,k)=(g,h)-(f,S\psi)=(g,h)-(S^*f,\psi)$$

for all $\{h,\psi\}\in T$, which implies $\{S^*f,g\}\in T^*$, i.e., $\{f,g\}\in T^*S^*$. This proves (2.19).

To show the last assertion, note that $S\in[\mathfrak{H}]$ implies that $\{0,g\}\in T^*S^*$ if and only if $\{0,g\}\in T^*$. Hence T^*S^* is an operator if and only if T^* is an operator. Recall that T^* is an operator if and only if $\operatorname{dom}T$ is dense in $\mathfrak{H}$. □

3. Spectral properties of the associated operators

Let A be a linear operator in a Hilbert space $\mathfrak{H}$ such that (2.1) holds and let B and C be operators in $[\mathfrak{H}]$ which satisfy (1.1). Certain properties of the operators B and C carry over to the associated operators $\widehat{B}$ and $\widehat{C}$.

LEMMA 3.1. *Let A be a linear operator in a Hilbert space $\mathfrak{H}$ such that* (2.1) *holds. Let B and C be operators in* $[\mathfrak{H}]$ *satisfying* (1.1) *and let $\widehat{B}$ and $\widehat{C}$ be the associated operators in $[\mathfrak{H}_A]$. Then the following statements are equivalent:*

(i) *the operator $\widehat{B}$ is normal;*
(ii) *the operator $\widehat{C}$ is normal;*
(iii) *$ABCx=ACBx$, $x\in\operatorname{dom}A$.*

PROOF. The following identities are straightforward consequences of (1.1), (2.5), and (2.7):

$$\widehat{B}\widehat{B}^*(Ax)=\widehat{B}\widehat{C}(Ax)=\widehat{B}A(Cx)=ABCx,$$

$$\widehat{B}^*\widehat{B}(Ax)=\widehat{C}\widehat{B}(Ax)=\widehat{C}A(Bx)=ACBx,$$

for all $x\in\operatorname{dom}A$. Since $\operatorname{ran}A$ is dense in $\mathfrak{H}_A$, the equivalence of (i) and (iii) is now clear, The equivalence of (ii) and (iii) follows similarly. □

Under additional conditions the spectra of the associated operators $\widehat{B}$ and $\widehat{C}$ can be estimated in terms of the spectra of B and C.

THEOREM 3.2. *Let A be a linear operator in a Hilbert space $\mathfrak{H}$ such that* (2.1) *holds. Let B and C be operators in $[\mathfrak{H}]$ satisfying* (1.1) *and let $\widehat{B}$ and $\widehat{C}$ be the associated operators in $[\mathfrak{H}_A]$. Assume in addition that the operators B and C satisfy*

(3.1)
$$(B-\lambda)^{-1}\operatorname{dom} A=\operatorname{dom} A,\quad (C-\lambda)^{-1}\operatorname{dom} A=\operatorname{dom} A,\quad \lambda\in\rho(B)\cap\rho(C).$$

Then the following spectral inclusions are valid:

$$\sigma(\widehat{B})\subset\sigma(B)\cup\sigma(C^*),\quad \sigma(\widehat{C})\subset\sigma(C)\cup\sigma(B^*). \tag{3.2}$$

PROOF. For any $\lambda\in\mathbf{C}$ it follows from (1.1) that

(3.3)
$$A(B-\lambda I)x=(C-\bar\lambda I)^*Ax,\quad A(C-\bar\lambda I)x=(B-\lambda I)^*Ax,\quad x\in\operatorname{dom} A.$$

In other words, $B-\lambda I$ and $C-\bar\lambda I$ may replace B and C in (1.1). Denote the corresponding associated operators in $[\mathfrak{H}_A]$ by $(B-\lambda I)^\wedge$ and $(C-\bar\lambda I)^\wedge$. They clearly satisfy

$$(B-\lambda I)^\wedge=\widehat{B}-\lambda\widehat{I},\quad (C-\bar\lambda I)^\wedge=\widehat{C}-\bar\lambda\widehat{I}.$$

Now assume that $\lambda\in\rho(B)\cap\rho(C^*)$ (so that $\lambda\in\rho(B)$ and $\bar\lambda\in\rho(C)$). Then it follows from (3.1) and (3.3) that

$$A(B-\lambda I)^{-1}x=(C-\bar\lambda I)^{-*}Ax,$$

$$A(C-\bar\lambda I)^{-1}x=(B-\lambda I)^{-*}Ax,\quad x\in\operatorname{dom} A.$$

Therefore, also the operators $(B-\lambda I)^{-1}$ and $(C-\bar\lambda I)^{-1}$ have associated operators in $[\mathfrak{H}_A]$, denoted by

$$\big[(B-\lambda I)^{-1}\big]^\wedge,\quad \big[(C-\bar\lambda I)^{-1}\big]^\wedge.$$

Observe that

$$\begin{aligned}(\widehat{B}-\lambda\widehat{I})\big[(B-\lambda I)^{-1}\big]^\wedge(Ax)&=(B-\lambda I)^\wedge A(B-\lambda I)^{-1}x\\&=A(B-\lambda I)(B-\lambda I)^{-1}x=Ax,\quad x\in\operatorname{dom} A,\end{aligned}$$

and, similarly,

$$\big[(B-\lambda I)^{-1}\big]^{\wedge}(\widehat{B}-\lambda\widehat{I})(Ax) = \big[(B-\lambda I)^{-1}\big]^{\wedge}A(B-\lambda I)x$$
$$= A(B-\lambda I)^{-1}(B-\lambda I)x = Ax, \quad x\in\operatorname{dom}A.$$

Since the range $\operatorname{ran}A$ is dense in $\mathfrak{H}_A$, these two identities show that

$$\big[(B-\lambda I)^{-1}\big]^{\wedge} = (\widehat{B}-\lambda\widehat{I})^{-1},$$

and $\lambda\in\rho(\widehat{B})$. In a similar way it can be shown that

$$\big[(C-\bar{\lambda}I)^{-1}\big]^{\wedge} = (\widehat{C}-\bar{\lambda}\widehat{I})^{-1}.$$

and $\lambda\in\rho(\widehat{C})$. In other words, it has been shown that $\rho(B)\cap\rho(C^*)\subset\rho(\widehat{B})$, and that $\rho(B)\cap\rho(C^*)\subset\rho(\widehat{C})$, which are equivalent to the inclusions in (3.2). □

Conditions on the spectrum of B and C give rise to restrictions on the spectrum of $\widehat{B}$ and $\widehat{C}$. Hence, when it is known that $\widehat{B}$ and $\widehat{C}$ are normal, then due to the spectral theorem further conclusions about $\widehat{B}$ and $\widehat{C}$ can be drawn on the basis of their spectral behaviour.

THEOREM 3.3. *Let A be a linear operator in a Hilbert space $\mathfrak{H}$ such that* (2.1) *holds. Let B and C be operators in $[\mathfrak{H}]$ satisfying* (1.1) *and* (3.1)*, and let $\widehat{B}$ and $\widehat{C}$ be the associated operators in $[\mathfrak{H}_A]$. Assume that one of the equivalent conditions in Lemma* 3.1 *is satisfied. Then the following statements are valid:*

(i) *if $\sigma(B),\sigma(C)\subset\mathbf{R}$, then $J^{**}\widehat{B}J^*$ and $J^{**}\widehat{C}J^*$ are equal selfadjoint extensions in $[\mathfrak{H}]$ of the operators C^*A and B^*A, respectively;*

(ii) *if $\sigma(B),\sigma(C)\subset[0,\infty)$, then $J^{**}\widehat{B}J^*$ and $J^{**}\widehat{C}J^*$ are equal nonnegative extensions in $[\mathfrak{H}]$ of the operators C^*A and B^*A, respectively.*

PROOF. By Theorem 2.2 and Lemma 3.1 the operators $\widehat{B}$ and $\widehat{C}$ in $[\mathfrak{H}_A]$ are normal, and $\widehat{B}^*=\widehat{C}$.

(i) If $\sigma(B),\sigma(C)\subset\mathbf{R}$, then it follows from (3.2) that the normal operators $\widehat{B}$ and $\widehat{C}$ have real spectrum. Hence $\widehat{B}$ and $\widehat{C}$ are selfadjoint in $[\mathfrak{H}_A]$ and $\widehat{B}=\widehat{C}$. Therefore $J^{**}\widehat{B}J^*=J^{**}\widehat{C}J^*$ is a selfadjoint operator in $[\mathfrak{H}]$.

(ii) If $\sigma(B),\sigma(C)\subset[0,\infty)$, then it follows from (i) and (3.2) that the selfadjoint operator $\widehat{B}=\widehat{C}$ has a nonnegative spectrum. Hence $\widehat{B}=\widehat{C}$ is nonnegative in $[\mathfrak{H}_A]$ and $J^{**}\widehat{B}J^*=J^{**}\widehat{C}J^*$ is nonnegative in $[\mathfrak{H}]$. □

COROLLARY 3.4. *Let A be a linear operator in a Hilbert space $\mathfrak{H}$ such that* (2.1) *holds. Let B be an operator in $[\mathfrak{H}]$ satisfying*

$$B(\operatorname{dom} A) \subset \operatorname{dom} A, \quad B^*A = AB \restriction \operatorname{dom} A. \tag{3.4}$$

Then $\widehat{B}$ is selfadjoint in $[\mathfrak{H}_A]$ and $J^{**}\widehat{B}J^*$ is a selfadjoint extension in $[\mathfrak{H}]$ of B^*A. Assume in addition that

$$(B-\lambda)^{-1}\operatorname{dom} A = \operatorname{dom} A, \quad \lambda \in \rho(B). \tag{3.5}$$

If $\sigma(B) \cap (-\infty, 0) = \emptyset$, then $J^{**}\widehat{B}J^*$ is a nonnegative extension in $[\mathfrak{H}]$ of B^*A.

PROOF. Let $B \in [\mathfrak{H}]$ satisfy (3.4) and define $C = B$. Then the pair of operators B and C satisfies (1.1). The duality (2.7) implies that $\widehat{B}$ is selfadjoint in $[\mathfrak{H}_A]$ and $J\widehat{B}J^*$ is selfadjoint in $[\mathfrak{H}]$. With $C = B$ the condition (3.5) implies that (3.1) is satisfied. If $\sigma(B) \cap (-\infty, 0) = \emptyset$, it follows from the spectral inclusion (3.2) that $\sigma(\widehat{B}) \subset [0, \infty)$. $\square$

Under the conditions of Corollary 3.4 and with the assumption $\sigma(B) \cap (-\infty, 0) = \emptyset$, it follows that B^*A is a nonnegative operator in $\mathfrak{H}$ with nonnegative extension $J^{**}\widehat{B}J^*$ in $[\mathfrak{H}]$. It is not known how to describe $J^{**}\widehat{B}J^*$ among all nonnegative extensions in $[\mathfrak{H}]$ of B^*A.

A more special version of these results can be found in [17, Theorem 3.2]; see also [9], [10].

4. Consequences of the construction

In this section all operators will belong to $[\mathfrak{H}]$, and a number of results are collected which follow from the previous constructions. Observe that if A is a selfadjoint operator in $[\mathfrak{H}]$ and if there are operators B and C in $[\mathfrak{H}]$ such that

$$AB = C^*A, \tag{4.1}$$

then (1.1) and (3.1) are automatically satisfied with $\operatorname{dom} A = \mathfrak{H}$. Clearly, if (4.1) holds then $\operatorname{ran} A$ is invariant under both B^* and C^*. Conversely, if $\operatorname{ran} A$ with $A = A^* \in [\mathfrak{H}]$ is, for instance, invariant under B^* with $B \in [\mathfrak{H}]$, then there exists $C \in [\mathfrak{H}]$ such that (4.1) holds, cf. [5]. If A is a nonnegative operator in $[\mathfrak{H}]$ and if there are operators B and C in $[\mathfrak{H}]$ such that (4.1) holds, then the associated operators $\widehat{B}$ and $\widehat{C}$ satisfy $AB = J\widehat{B}J^*$ and $AC = J\widehat{C}J^*$.

THEOREM 4.1. *Let $A, B, C \in [\mathfrak{H}]$ and let A be nonnegative. Assume that the identities*

$$AB = C^*A, \quad ABC = ACB, \tag{4.2}$$

are satisfied. Then the following statements are valid:

(i) *if $AB \neq 0$, then $\sigma(B)$ or $\sigma(C)$ is nonzero;*

(ii) *if $\sigma(B), \sigma(C) \subset \mathbf{R}$, then AB is selfadjoint;*

(iii) *if $\sigma(B), \sigma(C) \subset [0, \infty)$, then AB is nonnegative.*

PROOF. (i) According to Lemma 3.1 the condition $ABC = ACB$ implies that $\widehat{B}$ is normal. If $AB \neq 0$, it follows from $AB = J\widehat{B}J^*$ that $\widehat{B}$ is a nonzero operator. Therefore $\sigma(\widehat{B})$ contains a complex number of modulus $\|\widehat{B}\| > 0$. It follows from Theorem 3.2 that this point also belongs to $\sigma(B)$ or to $\sigma(C^*)$.

(ii) If $\sigma(B), \sigma(C) \subset \mathbf{R}$, it follows from Theorem 3.2 that $\sigma(\widehat{B}) \subset \mathbf{R}$. Therefore the normal operator $\widehat{B}$ is in fact selfadjoint, so that also AB is selfadjoint.

(iii) If $\sigma(B), \sigma(C) \subset [0, \infty)$, it follows from Theorem 3.2 that $\sigma(\widehat{B}) \subset [0, \infty)$. Therefore the selfadjoint operator $\widehat{B}$ is in fact nonnegative, so that also AB is nonnegative. □

Associated with the situation described in Theorem 4.1 is the following definition which goes back to A.C. Zaanen [23] (see also [24]): an operator $B \in [\mathfrak{H}]$ is said to be *normalizable* (with respect to the nonnegative operator $A \in [\mathfrak{H}]$) if there exists an operator $C \in [\mathfrak{H}]$ for which the identities (4.2) hold. In this case, the operators $\widehat{B}$ and $\widehat{C}$ are normal and $\widehat{B}^* = \widehat{C}$.

COROLLARY 4.2. *Let $A, B \in [\mathfrak{H}]$ and let A be nonnegative. Assume that the identity*

$$AB = B^*A, \tag{4.3}$$

is satisfied. Then the following statements are valid:

(i) *if $AB \neq 0$, then $\sigma(B)$ contains nonzero real values;*

(ii) *if $\sigma(B) \cap (-\infty, 0) = \emptyset$, then AB is nonnegative.*

PROOF. Theorem 4.1 reduces to $\widehat{B}$ being selfadjoint, as $\widehat{B}^* = \widehat{C} = \widehat{B}$. Hence, $\sigma(\widehat{B})$ is real, since it is contained in $\sigma(B)$ and $\mathbf{R}$. On the other hand $\widehat{B}$ is nonzero if $AB \neq 0$. □

Associated with the situation described in Corollary 4.2 is the following definition which goes back to A. Garbe [6], J. Marty [11], A. J. Pell [12], W. T. Reid [14], and A. C. Zaanen [21]: an operator $B \in [\mathfrak{H}]$ is said to be *symmetrizable* (with respect to the nonnegative operator $A \in [\mathfrak{H}]$) if $AB = B^*A$.

5. Some further intertwining results

THEOREM 5.1. *Let $A \in [\mathfrak{H}]$ be a nonnegative operator and let $B, C \in [\mathfrak{H}]$ be operators, such that $AB = C^*A$. Then there exists a unique operator S in $[\mathfrak{H}]$ with the properties*

$$\ker A \subset \ker S, \quad A^{\frac{1}{2}}B = SA^{\frac{1}{2}}, \quad C^*A^{\frac{1}{2}} = A^{\frac{1}{2}}S.$$

PROOF. Define the operator U from $\operatorname{ran} A \subset \mathfrak{H}_A$ into $\mathfrak{H}$ by

$$U(Ax) = A^{\frac{1}{2}}x, \quad x \in \mathfrak{H}.$$

Clearly, U is an isometry and U^* satisfies

$$U^*(A^{\frac{1}{2}}x) = Ax \in \mathfrak{H}_A, \quad x \in \mathfrak{H}.$$

Let $\widehat{B}$ and $\widehat{C}$ be the associated operators in $[\mathfrak{H}_A]$ corresponding to B and C. Define the operator S by

$$S = U\widehat{B}U^*, \tag{5.1}$$

so that $S \in [\mathfrak{H}]$. Then the required identities are obtained from

$$SA^{\frac{1}{2}}x = U\widehat{B}U^*A^{\frac{1}{2}}x = U\widehat{B}(Ax) = UA(Bx) = A^{\frac{1}{2}}Bx, \quad x \in \mathfrak{H}, \tag{5.2}$$

and, via (2.7), from

$$S^*A^{\frac{1}{2}}x = U\widehat{B}^*U^*A^{\frac{1}{2}}x = U\widehat{C}(Ax) = UA(Cx) = A^{\frac{1}{2}}Cx, \quad x \in \mathfrak{H}. \tag{5.3}$$

The uniqueness of S follows from $\ker A = \ker A^{\frac{1}{2}} \subset \ker S$. □

In other words: if the nonnegative operator A intertwines the operators B and C^*, then the square root $A^{\frac{1}{2}}$ intertwines B and S, and C^* and S, respectively. This theorem extends a result of Dieudonné [4], see also [17, Theorem 3]. When $C = B$, so that S is selfadjoint, the corresponding result can be found in [17]. The reasoning in the proof of Theorem 5.1 follows the arguments in [17].

COROLLARY 5.2. *Let $A \in [\mathfrak{H}]$ be a nonnegative operator and let $B, C \in [\mathfrak{H}]$ be operators, such that the identities*

$$AB = C^*A, \quad ABC = ACB.$$

hold. Then S in Theorem 5.1 is normal and

$$A^{\frac{1}{2}}BC = A^{\frac{1}{2}}CB.$$

PROOF. The additional assumption $ABC = ACB$ implies, by Lemma 3.1, that the operators $\widehat{B}$ and $\widehat{C}$ are normal and that $\widehat{B}^* = \widehat{C}$. Since U is isometric, it follows from the definition (5.1) of S, that S is normal:

$$S^*S = U^*\widehat{B}^*U^*U\widehat{B}U^* = U^*\widehat{B}^*\widehat{B}U^*, \quad SS^* = U\widehat{B}U^*U\widehat{B}^*U^* = U\widehat{B}\widehat{B}^*U^*.$$

Moreover, in this case, the identities (5.2) and (5.3) lead to

$$S^*SA^{\frac{1}{2}} = S^*A^{\frac{1}{2}}B = A^{\frac{1}{2}}CB, \quad SS^*A^{\frac{1}{2}} = SA^{\frac{1}{2}}C = A^{\frac{1}{2}}BC.$$

Since S is normal the assertion follows. □

COROLLARY 5.3. *Let $A, B \in [\mathfrak{H}]$ be operators such that*

$$A \geqq 0, \quad AB + BA = 0.$$

Then $AB = 0$.

PROOF. Clearly $B^*BA = -B^*AB$, so that B^*BA is a nonpositive (self-adjoint) operator. The general identity $\sigma(B^*BA) \cup \{0\} = \sigma(BAB^*) \cup \{0\}$, implies that B^*BA has a nonnegative spectrum. Hence $B^*AB = 0$, so that by Theorem 5.1 (with $C^* = -B$) it follows that $B^*AB = A^{\frac{1}{2}}S^*SA^{\frac{1}{2}}$. Therefore $A^{\frac{1}{2}}B = SA^{\frac{1}{2}} = 0$, which implies the assertion. □

The result in this corollary can be found in V. Prokaj [13]. Another simple proof of Corollary 5.3 is due to L. Molnár (personal communication, 1999): the identities

$$A^2B = A(AB) = A(-BA) = (-AB)A = (BA)A = BA^2,$$

imply that $AB = BA$ (via the square root of a nonnegative operator, cf. [15]), so that $AB = 0$.

References

[1] T. Ando and K. Nishio, Positive selfadjoint extensions of positive symmetric operators, *Tôhoku Math. J.,* **22** (1970), 65–75.

[2] Yu. M. Arlinskiĭ, S. Hassi, H. S. V. de Snoo and Z. Sebestyén, On the class of extremal extensions of a nonnegative operator, *Oper. Theory: Adv. Appl.* (B. Sz.-Nagy memorial volume), **127** (2001), 41–81.

[3] S. J. Bernau, Products of positive operators, *Proc. Cambridge Phil. Soc.,* **76** (1974), 415–416.

[4] J. A. Dieudonné, Quasi-hermitian operators, *Proc. Inter. Symp. Linear Algebra* (Israel, 1960), Jerusalem Academic Press (Jerusalem, 1961), 115–122.

[5] R. G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.,* **17** (1966), 413–415.

[6] A. Garbe, Zur Theorie der Integralgleichung dritter Art, *Math. Ann.,* **76** (1915), 527–547.

[7] S. Hassi, M. M. Malamud and H. S. V. de Snoo, On Kreĭn's extension theory of nonnegative operators, *Math. Nachr.,* **274/275** (2004), 40–73.

[8] S. Hassi, A. Sandovici, H. S. V. de Snoo and H. Winkler, A general factorization approach to the extension theory of nonnegative operators and relations, in preparation.

[9] M. G. Kreĭn, Compact linear operators on functional spaces with two norms, *Zbirnik Praz' Instituta Matem. Akad. Nauk URSR,* **9** (1947), 104–129 (Ukrainian) (English translation: *Integral Equations Operator Theory,* **30** (1998), 140–162).

[10] P. D. Lax, Symmetrizable linear transformations, *Comm. Pure Appl. Math.,* **7** (1954), 633–647.

[11] J. Marty, Valeurs singulières d'une equation de Fredholm, *C.R. Acad. Sci. Paris,* **150** (1910), 1499–1502.

[12] A. J. Pell, Applications of biorthogonal systems of functions to the theory of integral equations, *Trans. Amer. Math. Soc.,* **12** (1911), 165–180.

[13] V. Prokaj, *Operator extensions on Hilbert space,* Ph.D. dissertation, Department of Applied Analysis, Eötvös Loránd University (Budapest, 1996).

[14] W. T. Reid, Symmetrizable completely continuous linear transformations in Hilbert space, *Duke Math. J.,* **18** (1951), 41–56.

[15] F. Riesz and B. Sz.-Nagy, *Functional Analysis,* Ungar (New York, 1955).

[16] Z. Sebestyén, Operator extensions on Hilbert space, *Acta Sci. Math. (Szeged),* **57** (1993), 233–248.

[17] Z. Sebestyén, Positivity of operator products, *Acta Sci. Math. (Szeged),* **66** (2000), 287–294.

[18] Z. Sebestyén and J. Stochel, Restrictions of positive self-adjoint operators, *Acta Sci. Math. (Szeged),* **55** (1991), 149–154.

[19] Z. Sebestyén and J. Stochel, On products of unbounded operators, *Acta Math. Hungar.,* **100** (2003), 105–129.

[20] E. P. Wigner, On weakly positive matrices, *Canad. J. Math.,* **15** (1963), 313–317.

[21] A. C. Zaanen, Über vollstetige symmetrische und symmetrisierbare Operatoren, *Nieuw Archief voor Wiskunde,* **22** (1947), 57–80.

[22] A. C. Zaanen, Normalisable transformations in Hilbert space and systems of linear integral equations, *Acta Math.,* **83** (1950), 197–248.

[23] A. C. Zaanen, *Linear Analysis,* North-Holland (Amsterdam, 1953).

[24] H. J. Zimmerberg, On normalizable transformations in Hilbert space, *Acta Math.,* **86** (1951), 86–88.

(Received January 5, 2004)

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF VAASA
P.O. BOX 700
65101 VAASA
FINLAND
E-MAIL: SHA@UWASA.FI

DEPARTMENT OF APPLIED ANALYSIS
EÖTVÖS LORÁND UNIVERSITY
PÁZMÁNY PÉTER SÉTÁNY 1/C
1117 BUDAPEST
E-MAIL: SEBESTY@CS.ELTE.HU

DEPARTMENT OF MATHEMATICS AND COMPUTING SCIENCE
UNIVERSITY OF GRONINGEN
P.O. BOX 800
9700 AV GRONINGEN
NEDERLAND
E-MAIL: DESNOO@MATH.RUG.NL