Banach J. Math. Anal. (2021) 16:12
https://doi.org/10.1007/s43037-021-00167-1

Tusi Mathematical Research Group

ORIGINAL PAPER



Spectral analysis of bounded operators on semi-Hilbertian spaces

Hamadi Baklouti[1] · **Sirine Namouri**[1]

Received: 21 August 2021 / Accepted: 6 December 2021 / Published online: 3 January 2022
© Tusi Mathematical Research Group (TMRG) 2021, corrected publication 2024

Abstract
Let H be a Hilbert space and let A be a positive bounded operator on H. An operator $T \in B(H)$ is said to be A-invertible if there exists $S \in B(H)$ such that $ATS = AST = A$. In this paper we develop spectral analysis in relation with this new notion of invertibility. We show that this concept is well compatible with the semi-Hilbertian structure of H generated by the "indefinite" metric operator A. As well as placing the classical notion of an invertible operator in an appropriate setting, this new notion seems to be interesting for studies in the framework of non-Hermitian quantum mechanics.

Keywords Positive operator · Semi-inner product · Unitary operator · Spectrum

Mathematics Subject Classification 47A05 · 46C05

1 Introduction

During the past few years, there was a strong renewal of interest in the theory of operators on semi-Hilbertian spaces, see for examples [1–3, 5, 6, 8, 10, 11, 16, 18]. Actually, this structure is rich and promising as a branch of functional analysis. Besides, it seems that the interest on semi-Hilbertian analysis stems from the fact that this theory provides a more general point of view regarding the definition of operators representing physical observables in the context of quantum mechanics (QM, for short). Indeed, the physical states of the quantum system are represented on a Hilbert space H endowed with a given inner product $\langle ., . \rangle$. Typically, the

Communicated by Catalin Badea.

✉ Hamadi Baklouti
h.baklouti@gmail.com

Sirine Namouri
sirinenamouri@gmail.com

[1] Faculty of Sciences of Sfax, Sfax University, Sfax, Tunisia

operators that represent the physical observables of the system should be self-adjoint with respect to the given inner product. This view is rather, somewhat restrictive. The theory of non-Hermitian QM implements a more general criterion consisting in defining a new inner product, with respect to which the considered operators are self-adjoint. The new inner product is defined using an appropriate operator A called metric operator so that $\langle x, y\rangle_A = \langle Ax, y\rangle$ for any $x, y \in H$. In the framework of quasi-Hermitian QM [12, 17], and pseudo-Hermitian QM [14] the metric operator A is invertible, self-adjoint and positive with respect to the reference inner product. In contrast, in the setting of indefinite metric QM [4], the underlying operator is unitary and self-adjoint but not necessarily positive. In the mathematical approach, the operator A is self-adjoint and positive with respect to the usual inner product of H, but, in general, it is considered to be not invertible. This fact motivates us to introduce an appropriate criterion of invertibility with respect to A. We say that an operator $T \in B(H)$ is A-invertible if there exists $S \in B(H)$ such that $ATS = AST = A$. Here $B(H)$ stands for the algebra of every-where defined bounded linear operators on H. We investigate several properties in relation with this notion, and we show the compatibility of this new criterion of invertibility with the new structure governed by the new inner product.

The contents of the paper are the following. In Sect. 2, we set up notation and terminology. We also give a brief description of the result by Douglas [9] which plays a relevant role in this paper. In Sect. 3, we introduce and develop the study of the class of A-invertible operators. We study various structural properties of this class of operators. In Sect. 4, we investigate the A-invertibility in the subalgebra of A-bounded operators. Section 5 is the main part of this paper, it is devoted to the study of the spectral properties, of A-bounded operators, in relation with the new criterion of invertibility.

2 Preliminary results and notations

In this section, we introduce some basic notations, definitions and results that are needed to prove our main results. Throughout, $(H, \langle ., .\rangle)$ denotes a non trivial complex Hilbert space and $B(H)$ is the space of all everywhere defined bounded linear operators on H. Given $T \in B(H)$, we use $R(T)$ for its range, $N(T)$ for its null-space, and T^* for its adjoint. In the entire text, $A \neq 0$ denotes a positive operator on H, that is $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In a natural way, A generates a positive semi-definite sesquilinear form given by $\langle x, y\rangle_A = \langle Ax, y\rangle$ for all $x, y \in H$. The semi-norm associated with $\langle ., .\rangle_A$ is given by $\|x\|_A = \sqrt{\langle x, x\rangle_A}$ for all $x \in H$. The reference vector space H endowed with $\langle ., .\rangle_A$ is called a semi-Hilbertian space. Further, $\langle ., .\rangle_A$ induces on $B(H)$ the following quantity:

$$\|T\|_A := \sup_{\substack{x \in \overline{R(A)} \\ x \neq 0}} \frac{\|Tx\|_A}{\|x\|_A}.$$

The set of all operators T for which $\|T\|_A$ is finite is

$$B^A(H) = \{T \in B(H) : \exists c > 0, \text{ such that } \|Tx\|_A \leq c\|x\|_A, \forall x \in \overline{R(A)}\}.$$

The orthogonal projection on $\overline{R(A)}$ and the Moore–Penrose pseudoinverse of A provide important tools in the study of operators on the semi-Hilbertian space generated by the positive operator A.

Definition 2.1 [7] The Moore–Penrose inverse of A, denoted by $A^\dagger$, is defined as the unique linear extension of $\tilde{A}^{-1}$ to $D(A^\dagger) = R(A) \oplus \overline{R(A)}^\perp$ with $N(A^\dagger) = R(A)^\perp$, where $\tilde{A}$ is the isomorphism $\tilde{A} := A|_{N(A)^\perp} : N(A)^\perp = \overline{R(A)} \longrightarrow R(A)$.

It holds that $A^\dagger A = \mathcal{P}$ and $AA^\dagger = \mathcal{P}|_{D(A^\dagger)}$, where we use $\mathcal{P}$ for the orthogonal projection on $\overline{R(A)}$.

For a detailed survey of the properties of the Moore–Penrose inverse we refer to [7]. The following proposition gives some characterizations for an operator T to be in $B^A(H)$.

Proposition 2.2 [2] *For $T \in B(H)$ the following conditions are equivalent*:

1. $T \in B^A(H)$.
2. $A^{1/2}T(A^{1/2})^\dagger$ *is bounded*.
3. $R(A^{1/2}T^*A^{1/2}) \subseteq R(A)$.

Moreover, if one of these conditions holds, then $\|T\|_A = \|A^{1/2}T(A^{1/2})^\dagger\|$.

The following theorem due to Douglas plays a central role in this text (see [9]). Some recent generalizations and applications of this theorem are given in the review paper by Moslehian et al. [15], see also [13].

Theorem 2.3 *Let T and $S \in B(H)$. The following conditions are equivalent*:

1. $R(S) \subseteq R(T)$.
2. *There exists a positive number $\lambda > 0$ such that $SS^* \leq \lambda TT^*$.*
3. *There exists $D \in B(H)$ such that $TD = S$.*

If one of these conditions holds, then there exists a unique operator $D \in B(H)$ such that:

$$TD = S, \quad R(D) \subseteq \overline{R(T^*)} = N(T)^\perp, \quad N(D) = N(S).$$

D is called the Douglas's solution of $TX = S$.

Recall that an operator $W \in B(H)$ is said to be an A-adjoint of $T \in B(H)$ if $\langle Tx, y\rangle_A = \langle x, Wy\rangle_A$ hold for all $x, y \in H$, or equivalently $AW = T^*A$. We use $B_A(H)$ for the set of all operators which have A-adjoints. If $T \in B_A(H)$, then there exists a distinguished A-adjoint of T, namely

$$T^\sharp = A^\dagger T^* A.$$

This adjoint corresponds to the Douglas's solution of the equation $AX = T^*A$ (see [3], Sect. 2 for more useful properties of $T^\sharp$).

The classes of A-isometric operators and A-unitary operators are introduced and studied by M. L. Arias et al in [3].

- An operator $T \in B(H)$ is said to be A-isometric if $\|Tx\|_A = \|x\|_A$ for all $x \in H$.
- An operator $U \in B(H)$ is said to be A-unitary if $U \in B_A(H)$ and $\|Ux\|_A = \|U^\sharp x\|_A = \|x\|_A$ for all $x \in H$.

The following lemma allows to connect with fundamental properties of the usual isometries and unitary operators.

Lemma 2.4

1. *$T \in B_A(H)$ is an A-isometry if and only if $AT^\sharp T = A$.*
2. *$U \in B_A(H)$ is A-unitary if and only if $AU^\sharp U = AUU^\sharp = A$.*

Proof

1. Observe that T is an A-isometry if and only if $T^*AT = A$. Since $AT^\sharp = T^*A$, then $T^*AT = AT^\sharp T$. Therefore, T is an A-isometry if and only if $AT^\sharp T = A$.
2. Notice that U is A-unitary if and only if both U and $U^\sharp$ are A-isometric. By applying the first statement, it turns out to say that $AU^\sharp U = A$ and $A(U^\sharp)^\sharp U^\sharp = A$.

   On one hand, according to ([3], Lemma 2.1), we have $(U^\sharp)^\sharp = \mathcal{P}U\mathcal{P}$. On the other hand, since $U \in B_A(H)$, it follows that $U(N(A)) \subseteq N(A)$ and, therefore, $A\mathcal{P}U\mathcal{P} = AU$. Hence, $A(U^\sharp)^\sharp U^\sharp = AUU^\sharp$.

□

3 *A*-invertible operators

In this section, we introduce the notion of an A-invertible operator.

Definition 3.1 An operator $T \in B(H)$ is said to be A-invertible if there exists $S \in B(H)$ such that

$$ATS = AST = A.$$

S is said an A-inverse of T.

Remark 3.2 We have $A^\dagger A = \mathcal{P} = (A^{1/2})^\dagger A^{1/2}$ (see Definition 2.1). Also, $A\mathcal{P} = A$ and $A^{1/2}\mathcal{P} = \mathcal{P}A^{1/2} = A^{1/2}$. Then both the following assertions characterize the notion of an A-invertible operator.

- $T \in B(H)$ has an A-inverse $S \in B(H)$ if and only if $A^{1/2}TS = A^{1/2}ST = A^{1/2}$ hold.
- $T \in B(H)$ admits an A-inverse $S \in B(H)$ if and only if $\mathcal{P}TS = \mathcal{P}ST = \mathcal{P}$ hold.

This last characterization will often be used rather than the initial definition.

We list below some immediate consequences.

1. An invertible operator is A-invertible.
2. If A is one to one with a closed range, then A-invertible operators are the usual invertible operators.
3. If T is A-invertible and S is an A-inverse of T, then S is A-invertible and T is an A-inverse of S.
4. Not every operator admits an A-inverse and, in case it has one, there can be many. For instance, let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in B(\mathbb{C}^2)$. We verify by simple calculation that both $S_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $S_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ are A-inverses of T.
5. If A is injective, then any $T \in B(H)$ has at most one A-inverse.
   To see this, let S_1 and S_2 be two A-inverses of T. Then $ATS_1 = AS_1T = A$ and $ATS_2 = AS_2T = A$. In particular, we get $AT(S_1 - S_2) = 0$. Since A is injective then so is T, and hence also AT is, which implies that $S_1 - S_2 = 0$.

The next Proposition, which motivates the above definition, follows immediately from Lemma 2.4.

Proposition 3.3 *An operator $U \in B_A(H)$ is A-unitary if and only if it is A-invertible and $U^\sharp$ is an A-inverse of U.*

Further, we have the following result.

Proposition 3.4 *Let $T \in B_A(H)$ be an A-isometry. Then the following assertions are equivalent*:

1. *T is A-unitary.*
2. *T is A-invertible.*
3. *$T^\sharp$ is A-invertible with an A-inverse in $B_{A^{1/2}}(H)$.*

Proof

- (1) ⇒ (2) and (1) ⇒ (3) follow from Proposition 3.3.

- (2) $\Rightarrow$ (1) : Let $S \in B(H)$ be such that $\mathcal{P}TS = \mathcal{P}ST = \mathcal{P}$. Since T is an A-isometry, then $\mathcal{P}T^{\sharp}T = \mathcal{P}$, it remains to prove that $\mathcal{P}TT^{\sharp} = \mathcal{P}$. First, observe that $N(A)$ is an invariant subspace of the operators T and $T^{\sharp}$. As a consequence, $\mathcal{P}T\mathcal{P} = \mathcal{P}T$ and $\mathcal{P}T^{\sharp}\mathcal{P} = \mathcal{P}T^{\sharp}$. Then

$$\mathcal{P}T^{\sharp} = \mathcal{P}T^{\sharp}\mathcal{P} = \mathcal{P}T^{\sharp}(\mathcal{P}TS) = (\mathcal{P}T^{\sharp}T)S = \mathcal{P}S.$$

  Hence, $\mathcal{P}TT^{\sharp} = \mathcal{P}T(\mathcal{P}T^{\sharp}) = \mathcal{P}T(\mathcal{P}S) = \mathcal{P}TS = \mathcal{P}$, as claimed.
- (3) $\Rightarrow$ (1) : Let $R \in B_{A^{1/2}}(H)$ be such that $\mathcal{P}T^{\sharp}R = \mathcal{P}RT^{\sharp} = \mathcal{P}$. On one hand, since T is an A-isometry, $\mathcal{P}T^{\sharp}T = \mathcal{P}$. On the other hand, we have $\mathcal{P}R\mathcal{P} = \mathcal{P}R$, given that $N(A)$ is an invariant subspace of R. It follows that

$$\mathcal{P}T = (\mathcal{P}RT^{\sharp})T = \mathcal{P}R(\mathcal{P}T^{\sharp}T) = \mathcal{P}R,$$

  which in turn yields that $\mathcal{P}TT^{\sharp} = \mathcal{P}$. This proves that T is A-unitary.

□

Proposition 3.5 *If $T \in B(H)$ is A-invertible and S is an A-inverse of T, then TS and ST are A-unitary operators.*

Proof Since $ATS = A$, then $TS \in B_A(H)$ and

$$(TS)^{\sharp} = A^{\dagger}(TS)^{*}A = A^{\dagger}(ATS)^{*} = A^{\dagger}A = \mathcal{P}.$$

Hence,

$$A(TS)(TS)^{\sharp} = A\mathcal{P} = A \text{ and } A(TS)^{\sharp}(TS) = A\mathcal{P}TS = A\mathcal{P} = A.$$

Thus, according to Proposition 3.3, TS is A-unitary.

Similarly, using that $AST = A$ yields that ST is A-unitary. □

In the next proposition, we investigate the A-invertibility of the A-adjoints of an A-invertible operator.

Proposition 3.6 *Let $T \in B_A(H)$ and $S \in B_A(H)$ and let T' and S' be A-adjoints of T and S, respectively. Then the following conditions are equivalent*:

1. *T is A-invertible and S is an A-inverse of T.*
2. *T' is A-invertible and S' is an A-inverse of T'.*

Proof We have $AT' = T^{*}A$ and $AS' = S^{*}A$. Then

$$AT'S' = T^{*}AS' = T^{*}S^{*}A = (AST)^{*}.$$

As a consequence, $AT'S' = A$ holds if and only if $AST = A$. Similarly, we get $AS'T' = A$ if and only if $ATS = A$. This proves the desired equivalence. □

We get the following useful characterization.

Proposition 3.7 *Let $T, S \in B(H)$ be such that $ATS = A$. Suppose that $N(AT) = N(A)$. Then S is an A-inverse of T and $N(AS) = N(A)$.*

Proof We have $ATS = A$. Then

$$\begin{aligned} x \in N(A) &\Leftrightarrow ATSx = Ax = 0 \\ &\Leftrightarrow Sx \in N(AT) = N(A) \\ &\Leftrightarrow ASx = 0. \end{aligned}$$

Therefore, $N(AS) = N(A)$. It remains to show that $\mathcal{P}ST = \mathcal{P}$. We have

$$\begin{aligned} x \in N(\mathcal{P}ST) &\Leftrightarrow Tx \in N(AS) = N(A) \\ &\Leftrightarrow x \in N(AT) = N(A). \end{aligned}$$

Then, $N(\mathcal{P}ST) = N(A)$. On the other hand, using that $S(N(A)) \subseteq N(A)$ together with $T(N(A)) \subseteq N(A)$ leads up to

$$(\mathcal{P}ST)^2 = \mathcal{P}ST.$$

Hence, $\mathcal{P}ST$ is a projection. So, every $x \in H$ can be written as $x = y + z$ with $y = \mathcal{P}STx$ and $z \in N(\mathcal{P}ST)$. Also, since $R(\mathcal{P}ST) \subseteq \overline{R(A)}$ and $N(\mathcal{P}ST) = N(A)$, it follows that $\mathcal{P}x = y$. Thus, $\mathcal{P}ST = \mathcal{P}$. □

4 A-invertibility in $B_{A^{1/2}}(H)$

The set of all A-bounded operators in $B(H)$ will be denoted by $B_{A^{1/2}}(H)$.

$$B_{A^{1/2}}(H) = \{T \in B(H) : \exists c > 0; \|Tx\|_A \leq c\|x\|_A, \forall x \in H\}.$$

Recall that $B_{A^{1/2}}(H)$ is a subalgebra of $B(H)$ containing properly $B_A(H)$. Notice that, if $T \in B_{A^{1/2}}(H)$ then $T(N(A)) \subseteq N(A)$.

Definition 4.1 An operator $T \in B_{A^{1/2}}(H)$ is said to be invertible in $B_{A^{1/2}}(H)$ if it has an A-inverse $S \in B_{A^{1/2}}(H)$.

Example Let H be finite dimensional and let A be a positive operator (matrix) on H. Decomposing $H = R(A) \oplus N(A)$ leads up to the formula $A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & 0 \end{array}\right)$ with A_1 invertible. Clearly $B_{A^{1/2}}(H) = \{M = \left(\begin{array}{c|c} M_1 & 0 \\ \hline M_2 & M_3 \end{array}\right)\}$. It follows easily that $M = \left(\begin{array}{c|c} M_1 & 0 \\ \hline M_2 & M_3 \end{array}\right)$ is A-invertible if and only if M_1 is invertible. The general picture of the A-inverses of M is $S = \left(\begin{array}{c|c} M_1^{-1} & 0 \\ \hline S_2 & S_3 \end{array}\right)$ which is always in $B_{A^{1/2}}(H)$.

The scheme in an infinite dimensional Hilbert space is not that simple. In the following example, we construct an operator $T \in B_{A^{1/2}}(H)$ which is A-invertible with an A-inverse not in $B_{A^{1/2}}(H)$.

Example Let $H = \ell^2(\mathbb{N})$ be the space of square-summable complex sequences, and consider the positive operator $A \in B(H)$, given by $A(x_0, x_1, \dots) = (0, x_1, \dots)$. Now, let T be the shift operator on H defined by $T(x_0, x_1, \dots) = (x_1, x_2, \dots)$. Observe that T is A-invertible with an A-inverse S defined by $S(x_0, x_1, \dots) = (0, x_0, \dots)$. We verify by a simple calculation that $T \in B_{A^{1/2}}(H)$. On the other hand, it can be checked that $S(N(A)) \nsubseteq N(A)$, which implies that $S \notin B_{A^{1/2}}(H)$.

The following theorem establishes a characterization of A-invertibility in $B_{A^{1/2}}(H)$.

Theorem 4.2 *An operator $T \in B_{A^{1/2}}(H)$ is invertible in $B_{A^{1/2}}(H)$ if and only if the following two conditions are satisfied*:

(1) *There exists $c > 0$ such that $\frac{1}{c}\|x\|_A \leq \|Tx\|_A \leq c\|x\|_A$ for all $x \in H$.*
(2) $R(AT) = R(A)$.

Proof Let $T \in B_{A^{1/2}}(H)$, then for all $x \in H$,

$$\|Tx\|_A \leq \|T\|_A \|x\|_A.$$

If further T has an A-inverse $S \in B_{A^{1/2}}(H)$, then by Proposition 3.5

$$\|x\|_A = \|STx\|_A \leq \|S\|_A \|Tx\|_A$$

for all $x \in H$. Thus, there exists $c > 0$ such that for all $x \in H$,

$$\frac{1}{c}\|x\|_A \leq \|Tx\|_A \leq c\|x\|_A.$$

Moreover, since $ATS = A$, then $R(AT) = R(A)$.

Conversely, since $R(AT) = R(A)$, then by Theorem 2.3 there exists $S \in B(H)$ such that $ATS = A$. Using the condition (1) we get $Tx \in N(A)$ if and only if $x \in N(A)$, which means that $N(AT) = N(A)$. It follows from Proposition 3.7 that S is an A-inverse of T. Again using the condition (1) we prove easily that S is A-bounded.

□

The next result follows immediately by applying the first condition in Theorem 4.2.

Corollary 4.3 *If an operator $T \in B_{A^{1/2}}(H)$ is invertible in $B_{A^{1/2}}(H)$ then all of the A-inverses of T are inside $B_{A^{1/2}}(H)$.*

Further, we get the following result.

Theorem 4.4 *Let $T \in B_{A^{1/2}}(H)$ be invertible in $B_{A^{1/2}}(H)$. If S and $S^{'}$ are two A-inverses of T, then*

$$\|S\|_A = \|S^{'}\|_A.$$

Proof Since $S^{'}$ is an A-inverse of T, then $AS^{'}T = A$. So, for all $x \in H$,

$$\begin{aligned}\|Sx\|_A^2 &= \langle ASx, Sx\rangle \\ &= \langle (AS^{'}T)Sx, Sx\rangle \\ &\leq \|S^{'}TSx\|_A \|Sx\|_A \\ &\leq \|S^{'}\|_A \|TSx\|_A \|Sx\|_A.\end{aligned}$$

Given that $ATS = A$, then $\|TSx\|_A = \|x\|_A$. Hence, for all $x \in H$,

$$\|Sx\|_A^2 \leq \|S^{'}\|_A \|x\|_A \|Sx\|_A,$$

which leads up to

$$\|Sx\|_A \leq \|S^{'}\|_A \|x\|_A. \tag{4.1}$$

By taking the supremum over all $x \in H$ with $\|x\|_A = 1$, we obtain

$$\|S\|_A \leq \|S^{'}\|_A.$$

By analogous argument, we get

$$\|S^{'}\|_A \leq \|S\|_A.$$

Therefore, $\|S^{'}\|_A = \|S\|_A$. □

Now, we investigate the A-invertibility of the product of two A-invertible operators.

Proposition 4.5 *Let $T \in B_{A^{1/2}}(H)$ and $T^{'} \in B(H)$ be A-invertible with A-inverses $S \in B(H)$ and $S^{'} \in B_{A^{1/2}}(H)$, respectively, then $TT^{'}$ is A-invertible with an A-inverse $S^{'}S$.*

Proof According to Remark 3.2, we have $\mathcal{P}TS = \mathcal{P}ST = \mathcal{P}$ and $\mathcal{P}T^{'}S^{'} = \mathcal{P}S^{'}T^{'} = P$. Thus, using that $T(N(A)) \subseteq N(A)$, we get

$$ATT^{'}S^{'}S = AT(\mathcal{P}T^{'}S^{'})S = AT\mathcal{P}S = ATS = A.$$

Using that $S^{'}(N(A)) \subseteq N(A)$, analogous arguments lead up to $AS^{'}STT^{'} = A$. Thus, $TT^{'}$ is A-invertible with an A-inverse $S^{'}S$. □

In particular, we get

Corollary 4.6 *If $T, T' \in B_{A^{1/2}}(H)$ are invertible in $B_{A^{1/2}}(H)$, then so is TT'.*

The situation becomes more comfortable if the operators are commuting.

Proposition 4.7 *Let T and $T' \in B_{A^{1/2}}(H)$ be commuting ($TT' = T'T$). Then the following conditions are equivalent*:

1. *T and T' are invertible in $B_{A^{1/2}}(H)$.*
2. *TT' is invertible in $B_{A^{1/2}}(H)$.*

Proof According to Corollary 4.6, if T and T' are invertible in $B_{A^{1/2}}(H)$ then so is TT'. Conversely, suppose that $TT'(= T'T)$ has an A-inverse $S \in B_{A^{1/2}}(H)$. Then $AT(T'S) = A$. We should show that $T'S$ is an A-inverse of T. According to Proposition 3.7 it remains to show that $N(AT) = N(A)$. On the first hand, if $x \in N(A)$ then $Tx \in N(A)$ because $T \in B_{A^{1/2}}(H)$, which means that $N(A) \subseteq N(AT)$. On the other hand, we have $\mathcal{P}ST'\mathcal{P}T = \mathcal{P}ST'T = \mathcal{P}$. Let $x \in N(AT)$. It follows that

$$Tx \in N(A) \Leftrightarrow \mathcal{P}Tx = 0 \Rightarrow (\mathcal{P}ST')\mathcal{P}Tx = 0 \Leftrightarrow \mathcal{P}ST'Tx = 0 \Leftrightarrow \mathcal{P}x = 0 \Leftrightarrow x \in N(A).$$

Finally, observe that $T'S \in B_{A^{1/2}}(H)$. By analogous arguments, we prove that TS ($\in B_{A^{1/2}}(H)$) is an A-inverse of T'. □

As an immediate result, we get

Proposition 4.8 *Let $T \in B_A(H)$ be an A-normal operator. Then the following conditions are equivalent*:

1. *T and $T^\sharp$ are A-invertible with A-inverses in $B_{A^{1/2}}(H)$.*
2. *$TT^\sharp$ is A-invertible with an A-inverse in $B_{A^{1/2}}(H)$.*

The following lemma is useful in the proof of the next proposition.

Lemma 4.9 *If $T \in B_{A^{1/2}}(H)$ has an A-inverse $S \in B_{A^{1/2}}(H)$, then we have*

$$AS^{n+1}T = AS^n \text{ and } AT^{n+1}S = AT^n \text{ for all } n \in \mathbb{N}.$$

Proof It is enough to show the first statement. For this, we shall proceed by induction. Already, we have $AST = A$. Suppose that $AS^{n+1}T = AS^n$ for some $n \geq 0$. Since $A^\dagger A = \mathcal{P}$, it follows that $\mathcal{P}S^{n+1}T = \mathcal{P}S^n$. So, using that $S(N(A)) \subseteq N(A)$, we get

$$AS^{n+2}T = AS\mathcal{P}S^{n+1}T = AS\mathcal{P}S^n = AS^{n+1}.$$

Thus, $AS^{n+1}T = AS^n$ for all $n \in \mathbb{N}$. □

Proposition 4.10 *If $T \in B_{A^{1/2}}(H)$ is A-invertible with an A-inverse $S \in B_{A^{1/2}}(H)$, then T^n is A-invertible with an A-inverse S^n for all $n \in \mathbb{N}^*$.*

Proof Follows by induction using the preceding lemma. □

5 *A*-spectrum of an *A*-bounded operator.

In all this section A is a closed range positive operator. For a given $T \in B_{A^{1/2}}(H)$ and $\lambda \in \mathbb{C}$ the operator $\lambda I - T \in B_{A^{1/2}}(H)$. Further, we get the following result.

Theorem 5.1 *Let $T \in B_{A^{1/2}}(H)$. Then for $|\lambda| > \|T\|_A$, the operator $\lambda I - T$ is invertible in $B_{A^{1/2}}(H)$.*

Proof By applying Proposition 2.2, $A^{1/2}T(A^{1/2})^\dagger$ is a bounded operator and

$$\|T\|_A = \|A^{1/2}T(A^{1/2})^\dagger\|.$$

On the other hand we have $(A^{1/2}T(A^{1/2})^\dagger)^* = (A^{1/2})^\dagger T^* A^{1/2}$. It follows that

$$\|T\|_A = \|(A^{1/2})^\dagger T^* A^{1/2}\|.$$

Hence, for all $x \in H$,

$$\|((A^{1/2})^\dagger(\lambda I - T)^* A^{1/2})x\| = \|(\bar{\lambda}P - (A^{1/2})^\dagger T^* A^{1/2})x\| \geq (|\lambda| - \|T\|_A)\|Px\|.$$

By applying Theorem 2.3, for $|\lambda| > \|T\|_A$, there exists $S \in B(H)$ such that

$$A^{1/2}(\lambda I - T)(A^{1/2})^\dagger S = P.$$

Then

$$A(\lambda I - T)(A^{1/2})^\dagger S A^{1/2} = A.$$

Since A is closed range then so is $A^{1/2}$, and therefore the operator $(A^{1/2})^\dagger$ and hence $(A^{1/2})^\dagger S A^{1/2}$ is bounded. In particular $R(A) = R(A(\lambda I - T))$. On the other hand, for all $x \in H$,

$$(|\lambda| - \|T\|_A)\|x\|_A \leq \|(T - \lambda I)x\|_A \leq (|\lambda| + \|T\|_A)\|x\|_A.$$

Since $\|T\|_A < |\lambda|$, Theorem 4.2 leads up to the result. □

The following Lemma which proof is simple is important in our investigation.

Lemma 5.2 *Let $T \in B_{A^{1/2}}(H)$. Then the following conditions are equivalent*:

(1) *$T\mathcal{P}$ is invertible in $B_{A^{1/2}}(H)$ with an A-inverse S.*
(2) *$\mathcal{P}T$ is invertible in $B_{A^{1/2}}(H)$ with an A-inverse S.*

(3) *T is invertible in $B_{A^{1/2}}(H)$ with an A-inverse S.*

As an application of Theorem 5.1 we show that the class of operators which are invertible in $B_{A^{1/2}}(H)$ is stable under small perturbations.

Theorem 5.3 *Let $T \in B_{A^{1/2}}(H)$ be invertible in $B_{A^{1/2}}(H)$ with an A-inverse S, and let $T' \in B_{A^{1/2}}(H)$ be such that $1 > \|T'S\|_A$. Then $T + T'$ is invertible in $B_{A^{1/2}}(H)$.*

Proof We have $\mathcal{P}TS = \mathcal{P}ST = \mathcal{P}$. Therefore,

$$(T + T')\mathcal{P} = T\mathcal{P} + T'\mathcal{P} = T\mathcal{P} + T'\mathcal{P}ST\mathcal{P} = (I + T'\mathcal{P}S)T\mathcal{P}.$$

Given that $1 > \|T'S\|_A = \|T'\mathcal{P}S\|_A$, Theorem 5.1 yields that $I + T'\mathcal{P}S$ is invertible in $B_{A^{1/2}}(H)$. On the other hand, since T is invertible in $B_{A^{1/2}}(H)$, it follows from Lemma 5.2 that $T\mathcal{P}$ is invertible in $B_{A^{1/2}}(H)$. Thus, applying Corollary 4.6 permits to conclude that $(T + T')P$ and hence $T + T'$ is invertible in $B_{A^{1/2}}(H)$. □

Definition 5.4 Let $T \in B_{A^{1/2}}(H)$.

(1) The set $\rho_A(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible in } B_{A^{1/2}}(H)\}$ is called the A-resolvent set of T.
(2) The set $\sigma_A(T) = \mathbb{C} \setminus \rho_A(T)$ is called the A-spectrum of T. Thus

$$\sigma_A(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible in } B_{A^{1/2}}(H)\}.$$

It follows immediately from Lemma 5.2 the following:

Lemma 5.5 *For $T \in B_{A^{1/2}}(H)$ we have*:

$$\sigma_A(T) = \sigma_A(\mathcal{P}T) = \sigma_A(T\mathcal{P}).$$

In the following theorem we show that the resolvent set is open and we prove the existence of an A-inverse of $\lambda I - T$ which depends analytically on the parameter $\lambda \in \rho_A(T)$.

Theorem 5.6 *Let $T \in B_{A^{1/2}}(H)$. The A-resolvent set $\rho_A(T)$ is open in $\mathbb{C}$. Moreover there is an analytic function $R : \rho_A(T) \to B(H)$ with range in $B_{A^{1/2}}(H)$, satisfying $R(\lambda)$ is an A-inverse of $\lambda I - T$ for all $\lambda \in \rho_A(T)$.*

Proof Pick $\lambda_0 \in \rho_A(T)$ so that $\lambda_0 I - T$ has an A-inverse $R_0 \in B_{A^{1/2}}(H)$, and let $\lambda \in \mathbb{C}$ be such that $|\lambda - \lambda_0| < \min(\frac{1}{\|R_0\|}, \frac{1}{\|R_0\|_A})$.

Since $\|(\lambda - \lambda_0)IR_0\|_A < 1$, Theorem 5.3 yields that $\lambda \in \rho_A(T)$, showing that $\rho_A(T)$ is open.

Given that $|\lambda - \lambda_0| < \frac{1}{\|R_0\|}$, it holds that

$$R(\lambda) = \sum_{0}^{\infty} (\lambda_0 - \lambda)^k R_0^{k+1} \in B(H).$$

Since $\mathcal{P} = \mathcal{P}(\lambda_0 I - T)R_0$, by elementary calculus, one can show that

$$\mathcal{P}(\lambda I - T)R(\lambda) = \mathcal{P}[(\lambda_0 I - T) + (\lambda - \lambda_0)I]R(\lambda) = \mathcal{P}.$$

Further, using that $\mathcal{P}R_0(\lambda_0 I - T) = \mathcal{P}$, together with $\mathcal{P}R_0 = \mathcal{P}R_0\mathcal{P}$, yields

$$\mathcal{P}R(\lambda)(\lambda I - T) = \mathcal{P}R(\lambda)[(\lambda_0 I - T) + (\lambda - \lambda_0)I] = \mathcal{P}.$$

Thus $R(\lambda)$ is an A-inverse of $\lambda I - T$. Moreover, since $\lambda \in \rho_A(T)$, then $\lambda I - T$ has an A-inverse in $B_{A^{1/2}}(H)$. Applying Corollary 4.3 leads up to $R(\lambda) \in B_{A^{1/2}}(H)$. □

Corollary 5.7 *For $T \in B_{A^{1/2}}(H)$, the A-spectrum $\sigma_A(T)$ is not void.*

Proof Suppose that $\sigma_A(T) = \emptyset$. Then according to Theorem 5.6, there is an entire function $R(.)$ satisfying $R(\lambda) \in B_{A^{1/2}}(H)$, for all $\lambda \in \mathbb{C}$, and

$$A(\lambda I - T)R(\lambda) = AR(\lambda)(\lambda I - T) = A.$$

The inequality $\|(\lambda I - T)x\|_A \geq (|\lambda| - \|T\|_A)\|x\|_A$ yields that, for $|\lambda| > \|T\|_A$,

$$\|R(\lambda)\|_A \leq \frac{1}{|\lambda| - \|T\|_A}.$$

Hence, for each $x, y \in H$, the complex valued function $r : \lambda \mapsto \langle AR(\lambda)x, y\rangle$ is entire, and for $|\lambda| > \|T\|_A$, it holds that

$$|\langle AR(\lambda)x, y\rangle| \leq \|R(\lambda)\|_A \|x\|_A \|y\|_A \leq \frac{1}{|\lambda| - \|T\|_A} \|x\|_A \|y\|_A.$$

Hence, by Liouville's theorem, $r(.) = 0$. This, in turn, means that $AR(.) = 0$, in contradiction with $AR(\lambda)(\lambda I - T) = A$. □

By applying Theorem 5.1 together with Theorem 5.6, we get

Proposition 5.8 *For $T \in B_{A^{1/2}}(H)$,*

(1) $\sigma_A(T) \subset \overline{B}(0, \|T\|_A)$ *(the closed ball centered at* 0 *with radius* $\|T\|_A$*).*
(2) $\sigma_A(T)$ *is a compact set of* $\mathbb{C}$.

A more precise localization of the A-spectrum arises from the introduction of the notion of spectral radius. Recall from [11] the following

Definition 5.9 For $T \in B_{A^{1/2}}(H)$, the A-spectral radius of T is given by:

$$r_{\sigma,A}(T) = \inf_n \|T^n\|_A^{1/n}.$$

If $r_{\sigma,A}(T) = 0$, then T is said to be A-quasinilpotent.

Observing that $r_{\sigma,A}(T) \leq \|T\|_A$, the following proposition improves the first result in Proposition 5.8.

Proposition 5.10 *Let $T \in B_{A^{1/2}}(H)$.*

(1) *If $\lambda \in \sigma_A(T)$, then $\forall n \in \mathbb{N}^*$ we have $\lambda^n \in \sigma_A(T^n)$.*
(2) *If $|\lambda| > r_{\sigma,A}(T)$, then $\lambda \in \rho_A(T)$.*

Proof For $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}^*$, we have

$$\begin{aligned}\lambda^n I - T^n =& (\lambda I - T)(\lambda^{n-1} I + \cdots + T^{n-1}) \\ =& (\lambda^{n-1} I + \cdots + T^{n-1})(\lambda I - T).\end{aligned}$$

(1) Suppose that $\lambda^n \in \rho_A(T^n)$ so that $\lambda^n I - T^n$ is invertible in $B_{A^{1/2}}(H)$. By using the fact that $B_{A^{1/2}}(H)$ is an algebra, Proposition 2.2 yields that $\lambda I - T$ is invertible in $B_{A^{1/2}}(H)$. Therefore, $\lambda \in \rho_A(T)$. This proves the first statement.
(2) Suppose that $\lambda \in \sigma_A(T)$. It follows from the first statement, that for all $n \in \mathbb{N}^*$, $\lambda^n \in \sigma_A(T^n)$. On the other hand, by Proposition 5.8, we have $\sigma_A(T^n) \subset \overline{B}(0, \|T^n\|_A)$. Therefore, for all $n \in \mathbb{N}^*$, we have $|\lambda^n| \leq \|T^n\|_A$ and hence $|\lambda| \leq \inf_n \|T^n\|_A^{1/n} = r_{\sigma,A}(T)$.

□

As an immediate result we get:

Corollary 5.11 *If $T \in B_{A^{1/2}}(H)$ is A-quasinilpotent, then $\sigma_A(T) = \{0\}$.*

Next, we describe the A-spectrum of any A-inverse of $T \in B_{A^{1/2}}(H)$.

Theorem 5.12 *Let $T \in B_{A^{1/2}}(H)$ be invertible in $B_{A^{1/2}}(H)$. If S is an A-inverse of T, then*

$$\sigma_A(S) = \left\{\lambda^{-1} \,:\, \lambda \in \sigma_A(T)\right\}.$$

Proof Let $\lambda \neq 0 \in \mathbb{C}$. We have $\mathcal{P}TS = \mathcal{P}ST = \mathcal{P}$, then

$$\mathcal{P}(\lambda^{-1} I - S) = \mathcal{P}(T - \lambda I)\lambda^{-1}\mathcal{P}S = \lambda^{-1}\mathcal{P}S\mathcal{P}(T - \lambda I).$$

According to Lemma 5.2, $\lambda^{-1}\mathcal{P}S$ is invertible in $B_{A^{1/2}}(H)$. It follows from Proposition 4.7 that $\mathcal{P}(\lambda^{-1} I - S)$ is invertible in $B_{A^{1/2}}(H)$ if and only if so is $\mathcal{P}(T - \lambda I)$.

In turn, this is equivalent to that $T - \lambda I$ is invertible in $B_{A^{1/2}}(H)$. Therefore, $\sigma_A(S) = \left\{\lambda^{-1} : \lambda \in \sigma_A(T)\right\}$. □

In the following theorem we establish that two equivalent operators in $B_{A^{1/2}}(H)$ have the same A-spectrum.

Theorem 5.13 *Let $V \in B_{A^{1/2}}(H)$ be A-invertible with an A-inverse $W \in B_{A^{1/2}}(H)$. Then for all $T \in B_{A^{1/2}}(H)$ we have*

$$\sigma_A(WTV) = \sigma_A(T).$$

Proof Suppose that $\lambda \notin \sigma_A(WTV)$. Then, $\lambda I - WTV$ is invertible in $B_{A^{1/2}}(H)$. On the other hand, since $PVW = P$, then

$$PV(\lambda I - WTV)W = PV(\lambda I - WTPV)W = P(\lambda I - T).$$

According to Lemma 5.2, PV is invertible in $B_{A^{1/2}}(H)$. Then, by Corollary 4.6, $P(\lambda I - T)$ and hence $\lambda I - T$ itself is invertible in $B_{A^{1/2}}(H)$, so that $\lambda \notin \sigma_A(T)$. Thus we get:

$$\sigma_A(T) \subset \sigma_A(WTV).$$

Now, by applying Lemma 5.5, together with what we have just proven, we infer that

$$\sigma_A(T) \subset \sigma_A(WTV) \subset \sigma_A(V(WTV)W) = \sigma_A(PVWTPVW) = \sigma_A(PTP) = \sigma_A(T).$$

This concludes the proof. □

In particular, if $T \in B_{A^{1/2}}(H)$ and $U \in B_A(H)$ is an A-unitary operator then $\sigma_A(U^\sharp TU) = \sigma_A(T)$. Next, we describe the A-spectrum of A-unitary and A-isometric operators.

Theorem 5.14 *Let $U \in B_A(H)$ be A-unitary. Then*

$$\sigma_A(U) \subseteq \left\{\lambda \in \mathbb{C} : |\lambda| = 1\right\}.$$

Proof Suppose that $U \in B_A(H)$ is A-unitary, then $\|U\|_A = \|U^\sharp\|_A = 1$. According to Proposition 5.8,

$$\sigma_A(U) \subset \overline{B}(0,1) \text{ and } \sigma_A(U^\sharp) \subset \overline{B}(0,1).$$

On the other hand by Proposition 3.3, U is A-invertible and $U^\sharp$ is an A-inverse of U. Applying Theorem 5.12 yields that

$$\sigma_A(U^\sharp) = \left\{\lambda^{-1} : \lambda \in \sigma_A(U)\right\}.$$

Let $\lambda \in \sigma_A(U)$ so that $\lambda^{-1} \in \sigma_A(U^\sharp)$. Then $|\lambda| \leq 1$, and $|\lambda^{-1}| \leq 1$. As a consequence, $\sigma_A(U) \subseteq \left\{\lambda \in \mathbb{C} : |\lambda| = 1\right\}$. □

Theorem 5.15 *Let $T \in B_A(H)$ be an A-isometry which is not A-unitary. Then*

$$\sigma_A(T) = \left\{\lambda \in \mathbb{C} : |\lambda| \leq 1\right\}.$$

Proof Let $\lambda \in \mathbb{C}$ be such that $|\lambda| < 1$. Since T is an A-isometry, then $\mathcal{P}T^\sharp T = \mathcal{P}$. So,

$$\mathcal{P}T^\sharp \mathcal{P}(\lambda I - T) = \mathcal{P}T^\sharp(\lambda I - T) = \mathcal{P}(\lambda T^\sharp - I). \tag{5.1}$$

Now, we shall prove that if $T^\sharp$ is not invertible in $B_{A^{1/2}}(H)$, then so is $\lambda I - T$. Indeed, if $\lambda I - T$ is A-invertible with an A-inverse $S \in B_{A^{1/2}}(H)$, then $\mathcal{P}(\lambda I - T)S = \mathcal{P}S(\lambda I - T) = \mathcal{P}$. Hence, by (5.1), we get

$$\mathcal{P}T^\sharp = \mathcal{P}T^\sharp\mathcal{P} = \mathcal{P}T^\sharp\mathcal{P}(\lambda I - T)S = \mathcal{P}(\lambda T^\sharp - I)S. \tag{5.2}$$

Since T is an A-isometry, then $\|T^\sharp\|_A = \|T\|_A = 1$, and therefore $\|\lambda T^\sharp\|_A < 1$. So by using Theorem 5.1, we have $I - \lambda T^\sharp$ is invertible in $B_{A^{1/2}}(H)$. Hence, by Lemma 5.2, $\mathcal{P}(I - \lambda T^\sharp)$ is invertible in $B_{A^{1/2}}(H)$. Since S is invertible in $B_{A^{1/2}}(H)$, it holds by using (5.2) and Corollary 4.6, that $\mathcal{P}T^\sharp$ is invertible in $B_{A^{1/2}}(H)$. Again by Lemma 5.2, we obtain that $T^\sharp$ is invertible in $B_{A^{1/2}}(H)$. As a consequence, if $T^\sharp$ is not invertible in $B_{A^{1/2}}(H)$, then so is $\lambda I - T$.

Given that T is an A-isometry but not A-unitary, it follows from Proposition 3.4, that $T^\sharp$ is not invertible in $B_{A^{1/2}}(H)$. Therefore $\lambda I - T$ is not invertible in $B_{A^{1/2}}(H)$, which means that $\lambda \in \sigma_A(T)$. We have thus proven that $\left\{\lambda \in \mathbb{C} : |\lambda| < 1\right\} \subset \sigma_A(T)$. By the closedness of $\sigma_A(T)$ (Proposition 5.8), it follows that $\left\{\lambda \in \mathbb{C} : |\lambda| \leq 1\right\} \subset \sigma_A(T)$. On the other hand, since T is an A-isometry, then $\|T\|_A = 1$. Hence, by Proposition 5.8, $\sigma_A(T) \subset \overline{B}(0, 1)$. Therefore, $\sigma_A(T) = \left\{\lambda \in \mathbb{C} : |\lambda| \leq 1\right\}$. □

References

1. Antoine, J.P., Trapani, C.: Partial inner product spaces, metric operators and generalized hermiticity. J. Phys. A Math. Theor. **46**, 025204 (2013). ((**Corrigendum, J. Phys. A: Math. Theor. 46, 329501 (2013)**))
2. Arias, M.L., Corach, G., Gonzalez, M.C.: Metric properties of projections in semi-Hilbertian spaces. Integral Equ. Oper. Theory **62**, 11–28 (2008)
3. Arias, M.L., Corach, G., Gonzalez, M.C.: Partial isometries in semi-Hilbertian spaces. Linear Algebra Appl. **428**, 1460–1475 (2008)
4. Azizov, T.Ya., Iokhvidov, I.S.: In: Dawson, E.R. (eds.) Linear Operators in Spaces with Indefinite Metric. Wiley, Chichester (1989)
5. Baklouti, H., Feki, K., Mahmoud, S.O.A.: Joint normality of operators in semi-Hilbertian spaces. Linear Multilinear Algebra **68**(4), 845–866 (2020)
6. Baklouti, H., Namouri, S.: Closed operators in semi-Hilbertian spaces. Linear Multilinear Algebra **1**, 1 (2021). https://doi.org/10.1080/03081087.2021.1932709
7. Alves Barata, J.C., Hussein, M.S.: The Moore–Penrose pseudoinverse: a tutorial review of the theory. Braz. J. Phys. **42**, 146–165 (2012)
8. Bermúdez, T., Saddi, A., Zaway, H.: A, m-isometries on Hilbert spaces. Linear Algebra Appl. **540**, 95–111 (2018)
9. Douglas, R.G.: On majorization, factorization, and range inclusion of operators on Hilbert space. Proc. Am. Math. Soc. **17**, 413–415 (1966)
10. Feki, K.: Some numerical radius inequalities for semi-Hilbert space operators. J. Korean Math. Soc. **58**(6), 1385–1405 (2021)

11. Feki, K.: Spectral radius of semi-Hilbertian space operators and its applications. Ann. Funct. Anal. **11**, 929–946 (2020)
12. Krejčiřik, D., Lotoreichik, V., Znojil, M.: The minimally anisotropic metric operator in quasi-Hermitian quantum mechanics. Proc. R. Soc. A **474**, 20180264 (2018)
13. Manuilov, V., Moslehian, M.S., Xu, Q.: Douglas factorization theorem revisited. Proc. Am. Math. Soc. **148**, 1139–1151 (2020). https://doi.org/10.1090/proc/14757
14. Mostafazadeh, A.: Pseudo-Hermitian representation of quantum mechanics. Int. J. Geom. Methods Mod. Phys. **7**, 1191–1306 (2010)
15. Moslehian, M.S., Kian, M., Xu, Q.: Positivity of 2×2 block matrices of operators. Banach J. Math. Anal. **13**(3), 726–743 (2019)
16. Moslehian, M.S., Xu, Q., Zamani, A.: Seminorm and numerical radius inequalities of operators in semi-Hilbertian spaces. Linear Algebra Appl. **591**, 299–321 (2020)
17. Scholtz, F.G., Geyer, H.B., Hahne, F.J.W.: Quasi-Hermitian operators in quantum mechanics and the variational principle. Ann. Phys. (NY) **213**, 74–101 (1992)
18. Xu, Q., Ye, Z., Zamani, A.: Some upper bounds for the A-numerical radius of 2×2 block matrices. Adv. Oper. Theory **6**(1), 1–13