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Spectral analysis relative to a positive operator and applications

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ABSTRACT

Given a positive operator A on a Hilbert space H, we give a precise description of the spectrum relative to A of bounded operators. We extend the results of [Baklouti H, Namouri S. Spectral analysis of bounded operators on semi-Hilbertian spaces. Banach J Math Anal. 2022;16(1):12; Baklouti H, Mabrouk M. A note on the A-spectrum of A- bounded operators. Oper Matrices. 2023;17(3):599–611] by relaxing the closedness condition on the range of A. Also, we prove, in the general context, that the spectrum relative to A of an A-self-adjoint operator lies on the real line. A fact that strengthens the idea of considering the notion of A-invertibility in the setting of non-Hermitian quantum mechanics. As applications, we describe the spectrum of some weighted translation operators on L^2 spaces. Also, we get the spectral mapping theorem for the spectrum relative to A of bounded operators. Moreover, we investigate the spectrum of the product of bounded operators.

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1. Introduction

Let $(H, \langle\cdot,\cdot\rangle)$ be a non-trivial Hilbert space and $B(H)$ be the set of bounded operators on H. Given a non-trivial positive operator $A \in B(H)$, we equip H with a positive sesquilinear form $\langle x, y\rangle_A = \langle Ax, y\rangle$ for all $x, y \in H$. This new form is semi-definite if A is not injective. $(H, \langle\cdot,\cdot\rangle_A)$ is called a semi-hilbertian space. As usual, we define the semi-norm of $x \in H$ as $\|x\|_A = \langle x, x\rangle_A^{1/2}$, which is rather called the A-norm of x. The A-norm of an operator T in $B(H)$ is defined as $\|T\|_A = \sup_{\|x\|_A=1} \|Tx\|_A$. If there exists $W \in B(H)$ such that $\langle Tx, y\rangle_A = \langle x, Wy\rangle_A$ for all $x, y \in H$, then W is called an A-adjoint of T. It is immediate that W is an A-adjoint of T if and only if $T^*A = AW$. Similarly $R \in B(H)$ is called $A^{1/2}$-adjoint of T if $T^*A^{1/2} = A^{1/2}R$.

Notice that, the A-norm of an operator $T \in B(H)$ may be infinite. T is said A-bounded if there is some $c > 0$ such that $\|Tx\|_A \leq c\|x\|_A$ for all $x \in H$. Another important difficulty is that an operator in $B(H)$ may not have an A-adjoint. We use $B_A(H)$ for the set of operators in $B(H)$ having A-adjoints. Similarly, $B_{A^{1/2}}(H)$ designates the set of operators having $A^{1/2}$-adjoints. By applying Douglas's factorizing theorem [1] we infer that $B_{A^{1/2}}(H)$ coincides

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with the set of A-bounded operators. Recall from Ref. [2] that $B_A(H)$ and $B_{A^{1/2}}(H)$ are two subalgebras of $B(H)$ satisfying $B_A(H) \subseteq B_{A^{1/2}}(H)$.

The notion of A-invertibility of bounded operators on semi-Hilbertian spaces, and the associated spectrum, were first introduced in Ref. [3]. An operator $T \in B(H)$ is said to be A-invertible if there is some $S \in B(H)$ such that $ATS = A = AST$. Some characterizations of A-invertibility in the subalgebra of A-bounded operators were established. Further, if A is a closed range and positive, some fundamental properties of the A-spectrum were derived, see also Ref. [4]. Since then, much other work has developed this theory. K. Feki in Ref. [5] gives an affirmative answer to the question posed in Ref. [3] regarding the connection between the notions of A-spectral radius and A-spectrum for A-bounded operators. Detailed studies and numerous properties of the A-spectral radius and A-numerical range of semi-Hilbertian operators are given in Refs [6, 7] by K. Paul et al., see also the monograph [8] by P. Bhunia et al. M. Mabrouk and A. Zamani, in Ref. [9], introduce and study the notion of a-spectrum in the context of a C^*-algebra $\mathfrak{A}$, where $a \in \mathfrak{A}$ is positive. They establish some a-spectral permanence properties in case a is well supported. In Ref. [10], the authors prove that if A is a closed range then the A-spectrum of a bounded operator T is the usual spectrum of $\alpha(T)$, where α is a morphism of Banach algebras. In Ref. [11], the authors treat the A-spectrum of A-bounded operator on von-Neumann algebras.

In the present paper, we prove a new characterization of A-invertibility. We give a precise description of the A-spectrum of A-bounded operators. We treat the general case, where A may have an unclosed range. More precisely, let $T \in B_{A^{1/2}}(H)$ and $L \in B_{A^{1/2}}(H)$ be such that $T^*A^{1/2} = A^{1/2}L$. We show in Theorem 3.1 that $\sigma_A(T) = \sigma(T_{\text{eff}}) \cup \sigma(L^*_{\text{eff}})$ where T_{eff} and L_{eff} stand for the effective actions of T and L with respect to A (see Definition 2.2). Next, we prove in Theorem 4.1 that the A-spectrum of an A-self-adjoint operator lies on the real line. Recall that an operator is said A -self-adjoint, if it is an A-adjoint of itself. Such a result is fully in line with the fact that observables in quantum mechanics should have real spectra. This enhances the proposed idea of considering the notion of A-invertibility in the setting of (semi-definite) non-Hermitian quantum mechanics.

In the last section of this work, we introduce three applications. First, we investigate the spectra of a class of weighted translation operators defined on $L^2(\mathbb{R}, \mathrm{d}x)$. In quantum mechanics, a translation operator is defined as an operator which shifts particles and fields by a certain amount in a certain direction. This class of operators has been initially investigated by Parrott [12]. Many other researchers have been interested in studying these operators (see for example Refs [13–15]). Next, we address the application to the spectral mapping theorem. Notice that spectral mapping theorems are of practical importance. Especially, they are quite useful in describing quantum system under evolution, see Ref. [16]. Finally, we present an application of the previous results to describe the A-spectrum of the product of operators.

2. Preliminaries results

In this section we will recall some useful properties and refine some results in connection with the notion of A-invertibility. We arrive at a new characterization of A-invertible operators, see Proposition 2.3.

Let $(A^{1/2})^\dagger$ be the Moore-Penrose pseudo-inverse of $A^{1/2}$, that is the operator defined on the domain $D((A^{1/2})^\dagger) = N(A) \oplus R(A^{1/2})$ satisfying $(A^{1/2})^\dagger A^{1/2} = \mathcal{P}$ and

$A^{1/2}(A^{1/2})^\dagger = \mathcal{P}|_{D((A^{1/2})^\dagger)}$. Here $\mathcal{P}$ stands for the orthogonal projection on $\overline{R(A)}$. Observe that $(A^{1/2})^\dagger$ is closed and densely defined. Proposition 2.3 in Ref. [17] states that if $T \in B_{A^{1/2}}(H)$, then $A^{1/2}T(A^{1/2})^\dagger$ is bounded on $D((A^{1/2})^\dagger)$. We use $\overline{A^{1/2}T(A^{1/2})^\dagger}$ for the unique continuous extension of $A^{1/2}T(A^{1/2})^\dagger$ to all of H. We get the following lemma, which gives an alternative proof of $B_A(H) \subseteq B_{A^{1/2}}(H)$ and provides more information on this inclusion.

Lemma 2.1: *Let $T \in B_A(H)$. Then $\overline{A^{1/2}T(A^{1/2})^\dagger} \in B_{A^{1/2}}(H)$. Moreover, if $W \in B_A(H)$ is an A-adjoint of T then $\overline{A^{1/2}W(A^{1/2})^\dagger}$ is $A^{1/2}$-adjoint of T.*

Proof: Let $W \in B_A(H)$ be an A-adjoint of T so that $AT = W^*A$. We have

$$(\overline{A^{1/2}T(A^{1/2})^\dagger})^* = (A^{1/2}T^*(A^{1/2})^\dagger)^* = (A^{1/2})^\dagger T^* A^{1/2}.$$

Let $y = A^{1/2}x \in R(A^{1/2})$. It follows that,

$$\begin{aligned}
(\overline{A^{1/2}T(A^{1/2})^\dagger})^* y &= (A^{1/2})^\dagger T^* A^{1/2} A^{1/2} x \\
&= (A^{1/2})^\dagger T^* A x \\
&= (A^{1/2})^\dagger A W x \quad (\text{since } T^*A = AW) \\
&= (A^{1/2})^\dagger A^{1/2} A^{1/2} W x \\
&= \mathcal{P} A^{1/2} W x \quad (\text{since } (A^{1/2})^\dagger A^{1/2} = \mathcal{P}) \\
&= A^{1/2} W x \in R(A^{1/2}) \quad (\text{since } \mathcal{P} A^{1/2} = A^{1/2}).
\end{aligned}$$

Thus $(\overline{A^{1/2}T(A^{1/2})^\dagger})^*(R(A^{1/2})) \subseteq R(A^{1/2})$ and hence $\overline{A^{1/2}T(A^{1/2})^\dagger} \in B_{A^{1/2}}(H)$.

Moreover, for $x = x_0 + A^{1/2}x_1 \in N(A) + R(A^{1/2}) = D((A^{1/2})^\dagger)$, it holds that

$$A^{1/2}(A^{1/2}W(A^{1/2})^\dagger)x = A^{1/2}(A^{1/2}W(A^{1/2})^\dagger)A^{1/2}x_1 = AW\mathcal{P}x_1 = T^*Ax_1 = T^*A^{1/2}x.$$

Since $D((A^{1/2})^\dagger)$ is dense in H, by the uniqueness of the continuous extension, it follows that $A^{1/2}\overline{A^{1/2}W(A^{1/2})^\dagger} = T^*A^{1/2}$. ■

Definition 2.1 ([3]): An operator $T \in B_{A^{1/2}}(H)$ is A-invertible in $B_{A^{1/2}}(H)$ if there exists $S \in B_{A^{1/2}}(H)$ such that $ATS = A = AST$. Such S is called an A-inverse of T. If no confusion may arise we will simply say T is invertible in $B_{A^{1/2}}(H)$.

The following characterization has shown to be very useful in the analysis of this notion of invertibility.

Proposition 2.1 ([3]): *An operator $T \in B_{A^{1/2}}(H)$ is invertible in $B_{A^{1/2}}(H)$ if and only if the following two conditions are satisfied:*

(1) *There exists $c > 0$ such that $(1/c)\|x\|_A \leq \|Tx\|_A \leq c\|x\|_A$ for all $x \in H$.*
(2) $R(AT) = R(A)$.

The next result establishes a new characterization of A-invertibility in $B_{A^{1/2}}(H)$.

Proposition 2.2: *Let $T \in B_{A^{1/2}}(H)$ and $L \in B(H)$ be such that $T^*A^{1/2} = A^{1/2}L$. Then T is invertible in $B_{A^{1/2}}(H)$ if and only if there is $c > 0$ such that $\forall x \in H$,*

(1) $c\|x\|_A \leq \|Tx\|_A$
(2) $c\|x\|_A \leq \|Lx\|_A$.

Proof: We have $\|Tx\|_A \leq \|T\|_A\|x\|_A$ for all $x \in H$. Then, according to Proposition 2.1, it turns out to prove that the condition (2) in Proposition 2.2 is equivalent to $R(A) \subseteq R(AT)$. In fact $c\|x\|_A \leq \|Lx\|_A$ for all $x \in H$ means that $c^2A^{1/2}A^{1/2} \leq (A^{1/2}L)^*(A^{1/2}L)$, which is equivalent to $R(A^{1/2}) \subseteq R((A^{1/2}L)^*) = R(A^{1/2}T)$ (by Douglas's factorizing theorem). Obviously, $R(A^{1/2}) \subseteq R(A^{1/2}T)$ would imply that $R(A) \subseteq R(AT)$. It remains to show the converse implication. Let $y = A^{1/2}x \in R(A^{1/2})$. Then $A^{1/2}y \in R(A) \subseteq R(AT)$ so that $A^{1/2}y = ATz$ for some $z \in H$. Hence

$$y = \mathcal{P}y = (A^{1/2})^\dagger A^{1/2}y = (A^{1/2})^\dagger ATz = \mathcal{P}A^{1/2}Tz = A^{1/2}Tz \in R(A^{1/2}T).$$

■

The notion of effective action with respect to a positive operator is first introduced in Ref. [10].

Definition 2.2: For $T \in B(H)$, we call the effective action of T with respect to A, the operator in $B(\overline{R(A)})$ given by $T_{\text{eff}}(x) = \mathcal{P}T(x)$, for all $x \in \overline{R(A)}$.

As immediate properties we get:

Lemma 2.2: • *For all $T \in B(H)$, $(T^*)_{\text{eff}} = (T_{\text{eff}})^*$.*
• *For $T, S \in B_{A^{1/2}}(H)$, $(TS)_{\text{eff}} = T_{\text{eff}}S_{\text{eff}}$.*
• *If A is one to one then $T_{\text{eff}} = T$.*

Also, we recall from Ref. [10] the following lemma.

Lemma 2.3 ([10]): *Let $T \in B_{A^{1/2}}(H)$. If T is invertible in $B_{A^{1/2}}(H)$ then T_{eff} is invertible in $B(\overline{R(A)})$. Moreover, if S is an A-inverse of T in $B_{A^{1/2}}(H)$, then $S_{\text{eff}} = T_{\text{eff}}^{-1}$.*

Notice that the converse of the above lemma is false in general. The following proposition establishes a more precise description.

Proposition 2.3: *Let $T \in B_{A^{1/2}}(H)$ and $L \in B(H)$ be such that $T^*A^{1/2} = A^{1/2}L$. Then T is invertible in $B_{A^{1/2}}(H)$ if and only if T_{eff} and L_{eff} are both invertible.*

Proof: According to Proposition 2.2, T is invertible in $B_{A^{1/2}}(H)$ if and only if there is $c > 0$ such that $\|Tx\|_A \geq c\|x\|_A$ and $\|Lx\|_A \geq c\|x\|_A$ for all $x \in H$.

We claim that:

L_{eff} is invertible if and only if $\exists c > 0$ such that $\forall x \in H, \|Tx\|_A \geq c\|x\|_A$.

Indeed,

$$\begin{aligned}
\|Tx\|_A \geq c\|x\|_A,\ \forall x \in H \quad &\Leftrightarrow \\
\|A^{1/2}Tx\| \geq c\|A^{1/2}x\|,\ \forall x \in H \quad &\Leftrightarrow \\
\|L^*A^{1/2}x\| \geq c\|A^{1/2}x\|,\ \forall x \in H \quad &\Leftrightarrow \\
\|L^*y\| \geq c\|y\|,\ \forall y \in R(A^{1/2}) \quad &\Leftrightarrow \\
\|L^*Py\| = \|L^*y\| \geq c\|y\|,\ \forall y \in \overline{R(A)}.&
\end{aligned}$$

The last inequality means that L^*P is invertible as an operator : $\overline{R(A)} \to \overline{R(A)}$. Passing to the adjoint ensures that $L_{\text{eff}} : \overline{R(A)} \to \overline{R(A)}$ is invertible. Thus the claim is proved. This, also, implies that T_{eff} is invertible if and only if there is $c > 0$ such that $\|Lx\|_A \geq c\|x\|_A$ for all $x \in H$. ■

Making use of the notion of effective action permits to prove the following permanence result.

Proposition 2.4: *If $T \in B_A(H)$ and $T^\sharp$ are both invertible in $B_{A^{1/2}}(H)$ then any A-inverse S of T is in $B_A(H)$. Moreover $S^\sharp$ is an A-inverse of $T^\sharp$.*

Proof: Since $AT^\sharp = T^*A$, then

$$A_{\text{eff}}(T^\sharp)_{\text{eff}} = T^*_{\text{eff}}A_{\text{eff}}.$$

T and $T^\sharp$ are invertible in $B_{A^{1/2}}(H)$. It follows, from Proposition 2.3, that T_{eff} and $(T^\sharp)_{\text{eff}}$ are both invertible. Hence T^*_{eff} and $(T^\sharp)^*_{\text{eff}}$ are also invertible. Thus

$$(T^*_{\text{eff}})^{-1}A_{\text{eff}} = A_{\text{eff}}((T^\sharp)_{\text{eff}})^{-1}.$$

Now, let S be an A-inverse of T. According to Lemma 2.3, the matrix form of S associated to the decomposition $H = N(A) \oplus \overline{R(A)}$ writes $S = \begin{bmatrix} S_1 & S_2 \\ 0 & (T_{\text{eff}})^{-1} \end{bmatrix}$, for some suitable operators S_1 and S_2. We get $S^*A = AY$ with $Y = \begin{bmatrix} 0 & 0 \\ 0 & ((T^\sharp)_{\text{eff}})^{-1} \end{bmatrix}$ which belongs to $B(H)$. Hence $S \in B_A(H)$.

For the second part of the proposition, we have $ATS = A = AST$ so that $S^*T^*A = A = T^*S^*A$. Using the identities $T^*A = AT^\sharp$ and $S^*A = AS^\sharp$ yields that $AS^\sharp T^\sharp = A = AT^\sharp S^\sharp$. ■

3. *A*-spectrum of an *A*-bounded operator

In this section we shall give a precise description of the A-spectrum of A-bounded operators. First, recall that in [3, Section 5], under the assumption that A is a closed range positive, it was proven that:

Proposition 3.1: *Let $A \in B(H)$ be closed range positive and $T \in B_{A^{1/2}}(H)$. Then for all $\lambda > \|T\|_A$, the operator $\lambda I - T$ is invertible in $B_{A^{1/2}}(H)$.*

In ([11, Example 2.1]), it was shown that, if A has an unclosed range this result may fail for some $T \in B_{A^{1/2}}(H)$. The following result extends the above proposition to the general context.

Proposition 3.2: *Let $T \in B_{A^{1/2}}(H)$. Then for all complex value λ such that $|\lambda| > \max\{\|T\|_A, \|\mathcal{P}T\|\}$, the operator $\lambda I - T$ is invertible in $B_{A^{1/2}}(H)$.*

Proof: For a given $\lambda \in \mathbb{C}$ and $x \in H$ we have,

$$(|\lambda| - \|T\|_A)\|x\|_A \leq \|(T - \lambda I)x\|_A \leq (|\lambda| + \|T\|_A)\|x\|_A. \tag{1}$$

Suppose that $|\lambda| > \max\{\|T\|_A, \|\mathcal{P}T\|\}$. In particular $\|T\|_A < |\lambda|$. According to Proposition 2.1 in combination with (1), all what remains to be proven is that $R(A(\lambda I - T)) = R(A)$. We rather prove that $R(A^{1/2}(\lambda I - T)) = R(A^{1/2})$ which would imply that $R(A(\lambda I - T)) = R(A)$.

Given that $T \in B_{A^{1/2}}(H)$ there is $S \in B_{A^{1/2}}(H)$ such that $A^{1/2}T = S^*A^{1/2}$. Hence

$$A^{1/2}(\lambda I - T) = (\lambda I - S^*)A^{1/2}. \tag{2}$$

It follows that $(\lambda I - S^*)(R(A^{1/2})) \subset R(A^{1/2})$. We shall prove that $\lambda I - S^*$ is an automorphism of $R(A^{1/2})$.

Let $\mathrm{R}(A^{1/2})$ denote the Hilbert space $R(A^{1/2})$ equipped with the inner product given by $(A^{1/2}\xi, A^{1/2}\eta) := < \mathcal{P}\xi, \mathcal{P}\eta >$ for every $\xi, \eta \in H$ (see [18, Section 2]). For $y = A^{1/2}x \in \mathrm{R}(A^{1/2})$ we have

$$\begin{aligned}
\|S^*y\|_{\mathrm{R}(A^{1/2})} &= \|S^*A^{1/2}x\|_{\mathrm{R}(A^{1/2})} \\
&= \|A^{1/2}Tx\|_{\mathrm{R}(A^{1/2})} \\
&= \|\mathcal{P}Tx\| = \|\mathcal{P}T\mathcal{P}x\| \\
&\leq \|\mathcal{P}T\|\|\mathcal{P}x\| \\
&\leq \|\mathcal{P}T\|\|A^{1/2}x\|_{\mathrm{R}(A^{1/2})} \\
&\leq \|\mathcal{P}T\|\|y\|_{\mathrm{R}(A^{1/2})}.
\end{aligned}$$

Thus, as an operator on $\mathrm{R}(A^{1/2})$, S^* satisfies $\|S^*\|_{\mathrm{R}(A^{1/2})} \leq \|\mathcal{P}T\|$.

Because $|\lambda| > \|\mathcal{P}T\|$, it follows that $\lambda I - S^*$ is an automorphism of $\mathrm{R}(A^{1/2})$. In particular $R(A^{1/2}) = R((\lambda I - S^*)A^{1/2})$. This in combination with (2) yields $R(A^{1/2}) = R(A^{1/2}(\lambda I - T))$. ■

Next, we introduce the spectrum associated with the notion of A-invertibility in $B_{A^{1/2}}(H)$.

Definition 3.1 ([3]): Let $T \in B_{A^{1/2}}(H)$.

(1) The set $\rho_A(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{is invertible in } B_{A^{1/2}}(H)\}$ is called the A-resolvent set of T.

(2) The set $\sigma_A(T) = \mathbb{C} \setminus \rho_A(T)$ is called the A-spectrum of T. Thus

$$\sigma_A(T) = \left\{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible in } B_{A^{1/2}}(H)\right\}.$$

Observe that for $T \in B_{A^{1/2}}(H)$, it holds that $AT\mathcal{P} = A\mathcal{P}T = AT$. It follows immediately that

$$\sigma_A(T) = \sigma_A(\mathcal{P}T) = \sigma_A(T\mathcal{P}). \tag{3}$$

Remark 3.1: In case A is one-to-one, A-invertible operators are simply the usual invertible ones. Consequently, if $T \in B_{A^{1/2}}(H)$ is invertible in $B_{A^{1/2}}(H)$ then it is invertible in the usual way. It follows that:

$$\text{If } A \text{ is one} - \text{to} - \text{one and } T \in B_{A^{1/2}}(H) \text{ then } \sigma(T) \subseteq \sigma_A(T). \tag{4}$$

Notice that this inclusion may be strict (see [11, Example 2.2]).

In fact, the property (4) is a particular case of the following general fact.

Proposition 3.3: *Let $T \in B_{A^{1/2}}(H)$ then $\sigma(\mathcal{P}T) \setminus \{0\} \subseteq \sigma_A(T)$.*

Proof: Let $\lambda \neq 0 \in \rho_A(T)$ so that there is $S \in B_{A^{1/2}}(H)$ satisfying

$$\mathcal{P}(T - \lambda I)S = \mathcal{P} = \mathcal{P}S(T - \lambda I).$$

Since $N(A)$ is an invariant subspace of both T and S then $\mathcal{P}T = \mathcal{P}T\mathcal{P}$ and $\mathcal{P}S = \mathcal{P}S\mathcal{P}$. It follows that

$$(\mathcal{P}T - \lambda I)\mathcal{P}S = \mathcal{P} = \mathcal{P}S(\mathcal{P}T - \lambda I).$$

Let $R \in B(H)$ be defined by

$$R(x) = \begin{cases} \mathcal{P}Sx, & \text{if } x = \mathcal{P}x \in \overline{R(A)} \\ -\dfrac{1}{\lambda}x, & \text{if } x \in N(A). \end{cases}$$

Write $y = (\mathcal{P}T - \lambda I)Rx$.

$$\begin{array}{ll} \text{If } x = \mathcal{P}x \in \overline{R(A)}, & \text{then } y = (\mathcal{P}T - \lambda I)\mathcal{P}Sx = \mathcal{P}x = x. \\ \text{If } x \in N(A) \text{ so that } \mathcal{P}x = 0, & \text{then } y = (\mathcal{P}T - \lambda I)(-\frac{1}{\lambda}x) = x. \end{array}$$

Hence $(\mathcal{P}T - \lambda I)R = I$. Similarly we get $R(\mathcal{P}T - \lambda I) = I$ proving that $\lambda \in \rho(\mathcal{P}T)$. ■

In the following theorem, using the notion of effective action, we give a precise description of the A-spectrum of an A-bounded operator in terms of usual spectra.

Theorem 3.1: *Let $T \in B_{A^{1/2}}(H)$ and $L \in B(H)$ be such that $T^*A^{1/2} = A^{1/2}L$. Then $\sigma_A(T) = \sigma(T_{\text{eff}}) \cup \sigma(L^*_{\text{eff}})$.*

Proof: For $\lambda \in \mathbb{C}$, we have $(T - \lambda I)^*A^{1/2} = A^{1/2}(L - \bar{\lambda}I)$. Applying Proposition 2.3 yields that $\lambda \in \rho_A(T)$ if and only if $\lambda \in \rho(T_{\text{eff}})$ and $\bar{\lambda} \in \rho(L_{\text{eff}})$. Thus $\sigma_A(T) = \sigma(T_{\text{eff}}) \cup \sigma(L^*_{\text{eff}})$. ■

Next, we list some immediate consequences.

(1) Suppose that A is one-to-one. Let $T, L \in B_{A^{1/2}}(H)$ be such that $T^*A^{1/2} = A^{1/2}L$. One gets $\sigma_A(T) = \sigma(T) \cup \sigma(L^*)$.
(2) Suppose that A and T commute. Then $\sigma_A(T) = \sigma(T_{\text{eff}})$.
Indeed: Since $AT = TA$ then $A^{1/2}T = TA^{1/2}$. Consequently $A^{1/2}L = T^*A^{1/2} = A^{1/2}T^*$. Therefore $A_{\text{eff}}^{1/2}L_{\text{eff}} = A_{\text{eff}}^{1/2}T_{\text{eff}}^*$. Using the injectivity of $A_{\text{eff}}^{1/2}$ yields that $L_{\text{eff}} = T_{\text{eff}}^*$ and hence $L_{\text{eff}}^* = T_{\text{eff}}$. Theorem 3.1 ensures that $\sigma_A(T) = \sigma(T_{\text{eff}})$.
(3) In particular if A is one-to-one and commutes with T then $\sigma_A(T) = \sigma(T)$.

Notice that for $T \in B_{A^{1/2}}(H)$ we have $R(T^*A^{1/2}) \subseteq R(A^{1/2}) \subseteq D((A^{1/2})^\dagger)$. This implies that the operator $L = (A^{1/2})^\dagger T^* A^{1/2}$ is bounded (see Ref. [17]). Moreover, L satisfies $A^{1/2}L = T^*A^{1/2}$. Further, we get the following consequence.

Corollary 3.1: *Suppose that* $R(T^*) \subset D((A^{1/2})^\dagger) = R(A^{1/2}) \oplus N(A)$. *Then*

$$\sigma_A(T) \setminus \{0\} = \sigma(T_{\text{eff}}) \cup \sigma(\mathcal{P}T) \setminus \{0\}.$$

Proof: Since $R(T^*) \subset D((A^{1/2})^\dagger) = R(A^{1/2}) \oplus N(A)$, it follows that $(A^{1/2})^\dagger T^*$ is bounded. Applying Jacobson's lemma yields that

$$\sigma(L) \setminus \{0\} = \sigma(A^{1/2}(A^{1/2})^\dagger T^*) \setminus \{0\} = \sigma(\mathcal{P}T^*) \setminus \{0\}.$$

Again, by using Jacobson's lemma one gets

$$\sigma(L^*) \setminus \{0\} = \sigma(T\mathcal{P}) \setminus \{0\} = \sigma(\mathcal{P}T) \setminus \{0\}.$$

On the other hand, since $L(\overline{R(A)}) \subseteq \overline{R(A)}$ and $L(N(A)) = \{0\}$, we infer that $\sigma(L_{\text{eff}}) \setminus \{0\} = \sigma(L) \setminus \{0\}$. Consequently $\sigma(L_{\text{eff}}^*) \setminus \{0\} = \sigma(L^*) \setminus \{0\}$ and therefore

$$\sigma(L_{\text{eff}}^*) \setminus \{0\} = \sigma(\mathcal{P}T) \setminus \{0\}.$$

■

Let $r_\sigma(T) = \inf_n \|T^n\|^{1/n}$ denote the spectral radius of $T \in B(H)$. We list below some immediate but important consequences of Theorem 3.1.

Corollary 3.2: *Let* $T \in B_{A^{1/2}}(H)$.

(1) *The A-resolvent set* $\rho_A(T)$ *is open in* $\mathbb{C}$.
(2) *The A-spectrum* $\sigma_A(T)$ *is not void.*
(3) $\sigma_A(T) \subset \mathbb{D}(0, \max\{r_\sigma(PL), r_\sigma(PT)\})$, *the closed complex disc.*
(4) $\sigma_A(T)$ *is a compact set of* $\mathbb{C}$.

Proof: For item (3), using Lemma 2.2 yields that $\|T_{\text{eff}}\| = \|PT\|$, $(T_{\text{eff}})^n = (T^n)_{\text{eff}}$ and $PT^n = (PT)^n$. It follows that $r_\sigma(T_{\text{eff}}) = r_\sigma(PT)$. The rest derives directly from Theorem 3.1. ■

Another direct consequence of Theorem 3.1 is the following result.

Proposition 3.4: *Let $T \in B_{A^{1/2}}(H)$ and $L \in B(H)$ be such that $T^*A^{1/2} = A^{1/2}L$. Then $\sigma_A(L) = \sigma_A^*(T) = \{\bar{\lambda} : \ \lambda \in \sigma_A(T)\}$.*

3.1. On the A-spectral radius of A-bounded operators

Recall from Ref. [19] that, for $T \in B_{A^{1/2}}(H)$, the A-spectral radius of T is given by:

$$r_{\sigma,A}(T) = \inf_n \|T^n\|_A^{1/n} = \lim_{n\to+\infty} \|T^n\|_A^{1/n}.$$

A natural quantity, in connection with the A-spectrum, is introduced in Ref. [3]. Namely

$$\widetilde{r_A}(T) = \sup\{|z| : z \in \sigma_A(T)\}.$$

It was proven in Ref. [3] that if A is a closed range,

$$r_{\sigma,A}(T) \geq \widetilde{r_A}(T). \tag{5}$$

Then, K. Feki showed, in Ref. [5], that the opposite inequality holds.

$$r_{\sigma,A}(T) \leq \widetilde{r_A}(T). \tag{6}$$

If A is not closed range, the inequality (6) still works. In contrast, (5) may fail, as it is shown in the following example.

Example 3.1: Consider the Hilbert space $H = \ell^2(\mathbb{N})$, the sequence $a_n = (1/2^n)_{n\geq 0}$ and the multiplication operator A defined on H by $A(x_n) = (a_n^2 x_n)$. Observe that A is one to one, and then $\mathcal{P} = I$. Let T and L be the shift operators given by

$$T(x_n) = 2\,(0, x_0, x_1, \ldots) \quad \text{and} \quad L(x_n) = (x_1, x_2, \ldots), \quad \text{for all } (x_n)_{n\geq 0} \in H.$$

One can check that $T^*A^{1/2} = A^{1/2}L$, which implies that $T, L \in B_{A^{1/2}}(H)$.

Further, $A^{1/2}T^n = (L^n)^*A^{1/2}$, for all $n \geq 0$, and therefore,

$$\|T^n\|_A = \|A^{1/2}T^n(A^{1/2})^\dagger\| = \|(L^n)^*A^{1/2}(A^{1/2})^\dagger\| = \|(L^n)^*\mathcal{P}\| = \|\mathcal{P}L^n\| = \|L^n\| = 1.$$

Thus

$$r_A(T) = 1.$$

On the other hand, given that A is one-to-one, $\sigma(T) = \sigma(T_{\text{eff}}) \subseteq \sigma_A(T)$. Thus

$$\widetilde{r_A}(T) \geq r_\sigma(T) = 2 > 1 = r_A(T).$$

Finally, observe that, Theorem 3.1 leads up to $\widetilde{r_A}(T) = \max(r_\sigma(T_{\text{eff}}), r_\sigma(L_{\text{eff}}))$. On the other hand, relying on ([20, Lemma 3.6]), $r_\sigma(T_{\text{eff}}) = r_A(L)$ and $r_\sigma(L_{\text{eff}}) = r_A(T)$. Hence, we get

$$\widetilde{r_A}(T) = \max(r_A(T), r_A(L)).$$

This result confirms Theorem 3.18 in Ref. [20].

4. *A*-spectrum of *A*-self-adjoint operators

An observable in quantum mechanics refers to a physical quantity that can be measured in a quantum system. Observables are represented by operators, acting on a Hilbert space which represents the state vectors of the system. The spectra of these operators represent the possible measurement outcomes, and therefore, must be real, even in the setting of non-hermitian quantum mechanics. The following result confirms this fact.

Theorem 4.1: *Let $T \in B_A(H)$ be A-self-adjoint, that is $T^*A = AT$. Then $\sigma_A(T)$ is contained in the real line.*

Proof: We want to show that $T - (\alpha + \mathrm{i}\beta)I$ is invertible in $B_{A^{1/2}}(H)$ for any real values α and $\beta \neq 0$. Up to replace T by $(1/\beta)(T - \alpha I)$, it suffices to do this for $T - \mathrm{i}I$.

We have $T \in B_A(H)$ and $T^*A = AT$. According to Lemma 2.1, we infer that $L := \overline{A^{1/2}T(A^{1/2})^\dagger} \in B_{A^{1/2}}(H)$, and $T^*A^{1/2} = A^{1/2}L$. Hence $(T - \mathrm{i}I)^*A^{1/2} = A^{1/2}(L + \mathrm{i}I)$. In view of Proposition 2.2, we may show that there is some $C > 0$ such that for all $x \in H$,

$$\begin{cases} \|(T - \mathrm{i}I)x\|_A \geq C\|x\|_A \\ \|(L + \mathrm{i}I)x\|_A \geq C\|x\|_A \end{cases}$$

For the first inequality, since $T^*A = AT$,

$$\begin{aligned} \|(T - \mathrm{i}I)x\|_A^2 &= \langle A(Tx - \mathrm{i}x), Tx - \mathrm{i}x\rangle \\ &= \langle ATx, Tx\rangle + \langle Ax, x\rangle + i\langle ATx, x\rangle - \mathrm{i}\langle T^*Ax, x\rangle \\ &= \langle ATx, Tx\rangle + \langle Ax, x\rangle \geq \langle Ax, x\rangle = \|x\|_A^2. \end{aligned}$$

Now, we shall prove the second inequality. First, observe that we do not lose generality by assuming that $R(A) = R(AT)$. Indeed, as $\sigma_A(T)$ is bounded, there is some real value λ such that $T - \lambda I$ is invertible in $B_{A^{1/2}}(H)$ and hence by Proposition 2.1, $R(A(T - \lambda I)) = R(A)$. Thus, Douglas's theorem leads up to $c^2A^2 \leq (AT)(AT)^* = (AT)^*(AT)$ for some $c > 0$. Up to multiplying T by a non-negative constant, we can take $c \geq 2$. Therefore $\|ATx\| \geq 2\|Ax\|, \forall x \in H$, and hence $2|\langle ATx, Ax\rangle| \leq \|ATx\|^2$. In particular

$$2|\mathrm{Im}\langle ATx, Ax\rangle| \leq \|ATx\|^2, \quad \forall x \in H \tag{7}$$

Let $x = x_0 + A^{1/2}x_1 \in D((A^{1/2})^\dagger) = N(A) + R(A^{1/2})$. By using the relations $(A^{1/2})^\dagger A^{1/2} = \mathcal{P}$, $A^{1/2}T\mathcal{P} = A^{1/2}T$, and $A^{1/2}x_0 = 0$, it follows that:

$$\begin{aligned} \|(L + \mathrm{i}I)x\|_A^2 &= \|A^{1/2}T(A^{1/2})^\dagger A^{1/2}x_1 + \mathrm{i}A^{1/2}x_1 + \mathrm{i}x_0\|_A^2 \\ &= \|ATx_1 - \mathrm{i}Ax_1\|^2 \\ &= \|ATx_1\|^2 + \|Ax_1\|^2 - 2\mathrm{Im}\langle ATx_1, Ax_1\rangle \\ (\text{By (7)}) &\geq \|Ax_1\|^2 = \|x\|_A^2 \end{aligned}$$

Thus $\|(L + \mathrm{i}I)x\|_A \geq \|x\|_A$ for all $x \in D((A^{1/2})^\dagger)$. As $D((A^{1/2})^\dagger)$ is dense in H this inequality holds true for all $x \in H$. ■

5. Applications

In this section, we give three important applications of the previous results. Namely, a description of the spectrum of weighted translation operators, a spectral mapping theorem and a relation between the A-spectra of the products of two operators.

5.1. Weighted translation operator

Let $(X, \mathcal{F}, \mu)$ be a measure space and let ψ be a bounded measurable function on X and $\tau : X \to X$ a measure-preserving transformation, that is $\tau^{-1}(F) \in \mathcal{F}$ and $\mu(\tau^{-1}(F)) = \mu(F)$, for all $F \in \mathcal{F}$. The bounded operator T_ψ defined on $L^2(X, \mathcal{F}, \mu)$ by the rule $T_\psi(f)(x) = \psi(x)f(\tau(x))$ is called a weighted translation. Here, we investigate the spectra of a class of weighted translations defined on $L^2(\mathbb{R}, \mathrm{d}x)$.

Theorem 5.1: *Let φ be a bounded real function satisfying*

(H1) $\psi(x) := |\varphi(x)/\varphi(x-1)|$ is bounded.
Consider T the weighted translation operator defined on $H = L^2(\mathbb{R})$ by the rule : $T_\psi(f)(x) = \psi(x)f(x-1)$. Then $\sigma(T_\psi) \subseteq \mathbb{D}(0,1)$, the closed complex unit disc. Suppose further that

(H2) for some integer $k \geq 1$ the sequence $(\int_n^{n+1} |\varphi(x)|^2 \,\mathrm{d}x / \int_n^{n+1} |\varphi(x-k)|^2 \,\mathrm{d}x)$ has a null cluster point.
Then

$$\sigma(T_\psi) = \mathbb{D}(0,1).$$

Before going forward we notice that as an example of φ one may consider the following function.

$$\varphi(x) = \begin{cases} \mathrm{e}^{-x^2/2} & \text{if } x \geq 0 \\ 1 & \text{if } x < 0. \end{cases}$$

Proof: Let A be the multiplication operator defined on H by $A(f)(x) = |\varphi(x)|^2 f(x)$ for all $f \in H$. Consider R the translation operator defined by the rule $R(f)(x) = f(x-1)$ for all $f \in H$. Then R is invertible and $R^{-1} = R^* = S$, the operator given by $S(f)(x) = f(x+1)$ for all $f \in H$.

Notice that R is in $B_{A^{1/2}}(H)$ but S is not. Indeed, $R^* A^{1/2} = A^{1/2} L$, where L is the operator given by $L(f)(x) = \psi(x+1)f(x+1)$. According to (H1), $L \in B(H)$. Also, by (H1), φ has no vanishing point, and therefore A is one-to-one. It follows that $R_{\mathrm{eff}} = R$ and $L_{\mathrm{eff}} = L$. Further, we get easily $L^* = T_\psi$. A straightforward calculus leads up to $\|R\|_A = \|L\| = 1$ and $\|\mathcal{P}R\| = 1$. It follows from Proposition 3.2 that $\sigma_A(R) \subset \mathbb{D}(0,1)$. By Theorem 3.1, $\sigma(R) \cup \sigma(T_\psi) = \sigma_A(R) \subseteq \mathbb{D}(0,1)$. We have thus proven the first assertion of the theorem.

To prove the second assertion, observe that as R is unitary, $\sigma(R)$ is inside the complex unit circle. Actually $\sigma(R)$ is the whole unit circle (see, for instance, Ref. [21]). We shall prove that $\sigma_A(R) = \mathbb{D}(0,1)$. According to Theorem 3.1, this would imply that $\sigma(T_\psi) = \mathbb{D}(0,1)$.

Pick $\lambda \neq 0 \in \mathbb{C}$ such that $|\lambda| < 1$. Then $R - \lambda I$ is invertible and

$$\begin{aligned}(R-\lambda I)^{-1} &= (R-\lambda SR)^{-1}\\ &= S(I-\lambda S)^{-1}\\ &= \frac{1}{\lambda}\sum_{k=1}^{\infty}(\lambda S)^k\end{aligned}$$

To show that $\sum_{k=1}^{\infty}(\lambda S)^k \notin B_{A^{1/2}}(H)$, consider f_n, the characteristic function of the interval $]n, n+1[$, for all $n \in \mathbb{Z}$. Then $Sf_n = f_{n-1}$ and hence $S^k f_n = f_{n-k}$. Thus

$$A^{1/2}\left(\sum_{k=1}^{\infty}(\lambda S)^k\right)\Big)f_n = \sum_{k=1}^{\infty}\lambda^k A^{1/2} f_{n-k}.$$

Observe that $\{A^{1/2}f_{n-k},\ k \geq 1\}$ are functions with disjoint supports. It follows that

$$\|\sum_{k=1}^{\infty}(\lambda S)^k f_n\|_A \geq |\lambda|^k \|f_{n-k}\|_A = |\lambda|^k \int_n^{n+1} |\varphi(x-k)|^2\,\mathrm{d}x.$$

On the other hand $\|f_n\|_A = \int_n^{n+1} |\varphi(x)|^2\,\mathrm{d}x$. Thus

$$\frac{\|\sum_{k=1}^{\infty}(\lambda S)^k f_n\|_A}{\|f_n\|_A} \geq |\lambda|^k \frac{\int_n^{n+1}|\varphi(x-k)|^2\,\mathrm{d}x}{\int_n^{n+1}|\varphi(x)|^2\,\mathrm{d}x}.$$

By (H2) there is some k such that the right-hand side of the last inequality is not uniformly bounded with respect to n. Hence $\sum_{k=1}^{\infty}(\lambda S)^k \notin B_{A^{1/2}}(H)$ proving that $\lambda \in \sigma_A(R)$. Consequently, $\mathbb{D}(0,1) \subseteq \sigma_A(R)$ and therefore $\sigma_A(R) = \mathbb{D}(0,1)$. ■

5.2. Spectral mapping theorem

As a second application of Theorem 3.1 we get the spectral mapping theorem for the A-spectrum of an A-bounded operator.

Theorem 5.2: *Let $T \in B_{A^{1/2}}(H)$ and $L \in B(H)$ be such that $T^*A^{1/2} = A^{1/2}L$. Let f be a holomorphic function in the complex disc $D(0,R)$. Assume that $\|T\| < R$ and $\|L\| < R$. Then $f(T) \in B_{A^{1/2}}(H)$. Moreover,*

$$\sigma_A(f(T)) = f(\sigma_A(T)).$$

Proof: Given that $\|T\| < R$ it follows that $f(T) \in B(H)$ (see [22, Section VII 3]). On the other hand, since we have $T^*A^{1/2} = A^{1/2}L$, it comes immediately that,

$$(T^n)^*A^{1/2} = (T^*)^nA^{1/2} = A^{1/2}L^n, \tag{8}$$

for all positive integer n. Let $\sum a_n z^n$ be the power series associated with f in $D(0,R)$ so that $f(T)$ is the uniform limit in $B(H)$ of $\sum_{n=0}^{N} a_n T^n$ as $N \to \infty$. Using (8) yields:

$$\begin{aligned} f(T)^*A^{1/2} &= \left(\sum_{n=0}^{+\infty} a_n T^n\right)^* A^{1/2} \\ &= \sum_{n=0}^{+\infty} \bar{a}_n (T^*)^n A^{1/2} \\ &= \sum_{n=0}^{+\infty} \bar{a}_n A^{1/2} L^n \\ &= A^{1/2} \sum_{n=0}^{+\infty} \bar{a}_n L^n \\ &= A^{1/2} \bar{f}(L). \end{aligned}$$

Since $\|L\| < R$ then $\bar{f}(L) \in B(H)$ which ensures that $f(T) \in B_{A^{1/2}}(H)$.

According to Theorem 3.1,

$$\sigma_A(f(T)) = \sigma(f(T)_{\text{eff}}) \cup \sigma((\bar{f}(L))^*_{\text{eff}}).$$

On the other hand,

$$f(T)_{\text{eff}} = f(T_{\text{eff}}).$$

Indeed, for all $x \in \overline{R(A)}$,

$$\begin{aligned} f(T)_{\text{eff}}(x) &= Pf(T)(x) \\ &= \mathcal{P} \sum_{n=0}^{+\infty} a_n T^n(x) \\ &= \sum_{n=0}^{+\infty} a_n \mathcal{P} T^n(x) \\ &= \sum_{n=0}^{+\infty} a_n (\mathcal{P}T)^n(x) \\ &= \sum_{n=0}^{+\infty} a_n (T_{\text{eff}})^n(x) = f(T_{\text{eff}})(x). \end{aligned}$$

Also,

$$(\bar{f}(L))^*_{\text{eff}} = (\bar{f}(L_{\text{eff}}))^* = f(L^*_{\text{eff}}).$$

Thus

$$\sigma_A(f(T)) = \sigma(f(T_{\text{eff}})) \cup \sigma(f(L^*_{\text{eff}})).$$

By using the spectral mapping theorem ([22, Section VII. 3. Theorem 11]) we get

$$\sigma_A(f(T)) = f(\sigma(T_{\text{eff}})) \cup f(\sigma(L^*_{\text{eff}})) = f(\sigma(T_{\text{eff}}) \cup \sigma(L^*_{\text{eff}})) = f(\sigma_A(T)).$$

■

5.3. A-spectrum of the product of operators

Jacobson's lemma states that the spectra of two products TS and ST of bounded operators coincide at all non-zero points. In this section, we shall establish a version of this important result for the A-spectrum.

Lemma 5.1: *Let $T, S \in B_{A^{1/2}}(H)$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$. Then $\lambda I - TS$ is invertible in $B_{A^{1/2}}(H)$ if and only if so is $\lambda I - ST$.*

Proof: Suppose that $\lambda I - TS$ has an A-inverse $R \in B_{A^{1/2}}(H)$ so that

$$\mathcal{P}(\lambda I - TS)R = \mathcal{P} = \mathcal{P}(\lambda I - TS)R.$$

Hence, using that $\mathcal{P}S\mathcal{P} = \mathcal{P}S$, we get

$$\begin{aligned}\mathcal{P}(\lambda I - ST)(SRT + I) &= \lambda\mathcal{P}SRT + \lambda\mathcal{P} - \mathcal{P}S(\mathcal{P}TSR)T - \mathcal{P}ST \\ &= \lambda\mathcal{P}SRT + \lambda\mathcal{P} - \mathcal{P}S\mathcal{P}(\lambda R - I)T - \mathcal{P}ST \\ &= \lambda P\end{aligned}$$

A similar computation leads up to $\mathcal{P}(SRT + I)(\lambda I - ST) = \lambda\mathcal{P}$. Since $SRT + I \in B_{A^{1/2}}(H)$ and $\lambda \neq 0$, it follows that $(\lambda I - ST)$ is invertible in $B_{A^{1/2}}(H)$. ■

An immediate consequence is the following result that extends the results of ([11, Corollary 2.6]) by relaxing the closedness condition on the range of A.

Corollary 5.1: *For $T, S \in B_{A^{1/2}}(H)$ it holds that $\sigma_A(TS) \setminus \{0\} = \sigma_A(ST) \setminus \{0\}$.*

The main result in Ref. [23] states that the product of two operators on a Hilbert space has a real spectrum if one of them is symmetric and the other is positive. Along this line, by combining Corollary 5.1 with Theorem 4.1, we get the following result.

Corollary 5.2: *Let $T, S \in B(H)$. Suppose that T is A-self-adjoint and $S = R^2$ where R is A-self-adjoint. Then $\sigma_A(TS)$ is contained in the real line.*

Proof: First, observe that T and S are both in $B_{A^{1/2}}(H)$. Applying Corollary 5.1 yields that

$$\sigma_A(TS) \setminus \{0\} = \sigma_A(TRR) \setminus \{0\} = \sigma_A(RTR) \setminus \{0\}.$$

On the other hand, since T and R are both A-self-adjoint, then so is RTR. The result follows from Theorem 4.1. ■

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